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STUDIES IN SEMANTICS

BY RUDOLF CARNAP

- I. Introduction to Semantics
- II. Formalization of Logic

S T U D I E S I N S E M A N T I C S
VOLUME II

FORMALIZATION OF LOGIC

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FORMALIZATION OF LOGIC

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PREFACE

The purpose of this book

In this volume, an application will be made of the method of semantics developed in *Introduction to Semantics*, the first volume of this series, *Studies in Semantics*. Among the characteristic features of this method are the distinction between a calculus and its interpretation, in other words, between syntactical and semantical systems; and the use of L-concepts based on the concept of L-range. The problem to be dealt with is that of the possibility and the scope of the formalization of logic. This problem has long been discussed, especially during the last hundred years, the period of the development of modern logic. However, before the method of semantics became available, no precise answer could be given, and not even a clear and precise formulation of the question was possible.

The task of the formalization of any theory, i.e. of its representation by a formal system or calculus, belongs to syntax, not to semantics. On the other hand, the question of whether a proposed calculus formalizes a given theory adequately and completely is a matter of the relations between a calculus and an interpreted system, and hence requires semantics in addition to syntax. In this book, the theory to be formalized is logic. Calculi representing logic in a formal way have been constructed and thoroughly investigated by many logicians. The most important and best known of these logical calculi are the propositional calculus (called PC in this book), containing the propositional connectives 'not', 'or', 'and', etc.; and, constructed on its basis, the functional calculus (here called FC), containing general sentences with terms like 'every' and 'there is'. Our problem will be to determine to what extent these calculi fulfill the task of formalizing

logic, and more generally, to what extent any calculus of the customary kind can fulfill this task. Contrary to the general belief, not all essential questions concerning PC have so far found their answer. For instance, the question whether PC completely formalizes all logical features of the part of logic covered by it, i.e. of the connectives, has not been answered by previous investigations. It seems to be the tacit assumption of many that this question is answerable in the affirmative. In this book, it will be shown that the answer is negative; and more generally, that no calculus of the customary kind can fulfill the task of a full formalization. However, a full formalization of propositional logic will be shown to be possible by making use of new concepts. A similar analysis of FC will be made, which leads to analogous negative results. And there likewise a full formalization of the logic of functions will be given by the construction of a new calculus. These results do not of course affect the value of the purely formal method of constructing 'calculi'; they rather make the foundations of that method more secure.

The role of semantics in the development of logic

Semantics — more exactly, pure semantics as here conceived — is not a branch of empirical science; it does not furnish knowledge concerning facts of nature. It is rather to be regarded as a tool, as one among the logical instruments needed for the task of getting and systematizing knowledge. As a hammer helps a man do better and more efficiently what he did before with his unaided hand, so a logical tool helps a man do better and more efficiently what he did before with his unaided brain, that is, by means of instinctive habits rather than through deliberate acts guided by explicit rules.

Aristotle's logic was the first logical tool of this kind. It did not originate the human activity of drawing inferences; from the time when language developed to the point of containing compound and general sentences, man has deduced conclusions from prem-

ises without once mentioning the mood *Barbara*. What was new in Aristotle's logic was not the activity but its systematization, that is, the construction of explicit rules for it. This made it possible to replace instinctive acts of inference by deliberate, methodical acts, and to examine critically the inferences made either instinctively or methodically.

The task of modern logic, as it has been developed since the middle of the last century, is fundamentally the same. The difference is only one of degree with respect to technical development, especially the multiplicity and efficiency of the tools. As a result of this development it has become possible not only to increase the safety and precision of the deductive method in realms already known, but also to reach results which could not have been obtained at all without the new tools. Although modern logic has already made a great advance in the degree of systematization and explicitness, nevertheless it has been long in reaching a full methodological understanding of its own procedures. This development of modern logic towards greater methodological consciousness is still going on and provides many of the basic problems for contemporary logical research.

Among the methodological tendencies or points of view in logic and especially in modern logic, two are of special interest for our present considerations. The one tendency emphasizes form, the logical structure of sentences and deductions, relations between signs in abstraction from their meaning. The other emphasizes just the factors excluded by the first: viz., meaning, interpretation; relations of entailment, compatibility, incompatibility, etc., as based on meaning; the distinction between necessary and contingent truth, etc. The two tendencies are as old as logic itself and have appeared under many names. Using contemporary terms, we may call them the syntactical and the semantical tendencies respectively. Theoretically they are not incompatible, but rather complementary to each other; yet in the historical development we find that logicians have sometimes emphasized one of

them at the sacrifice of the other. Usually, however, both points of view were combined without explicit distinction. It took many decades, even after modern logic was under way, before each of them was clearly recognized in its nature and represented by a pure method of its own. The formal, syntactical method was the first to be developed, and its emergence was stimulated by certain trends within mathematics, namely, the generalization of algebra, and the development of the postulational method especially in geometry. The elaboration of the formal method in logic is chiefly due to the works of Frege, Hilbert, and their followers. The main features of this method have often been described and discussed. The best description and analysis of its historical development has been given by Milton B. Singer in a study which will, I hope, soon be published. The development of the semantical method in a form clearly distinguished from the syntactical method is still in its first phases. Its origin in the Warsaw School of Logic and the first steps made by Tarski towards its systematization have been mentioned in the preface to Volume I of this series. Each of the two methods has the function of making systematic and explicit certain procedures which have been practically applied in traditional logic for the last two thousand years and, in a more elaborate and exact way, in modern logic for the last hundred years. Today it is generally recognized that the long-run tendency of gradually increasing formalization has found its necessary systematization in the modern syntactical method. In my opinion an analogous necessity prevails for a systematization of the long-run semantical tendency.

The decisive steps in the development of logic — e.g. Aristotle's syllogistic rules, Boole's creation of symbolical logical calculi, and the initiation of the syntactical method by Frege and Hilbert, to mention only a few outstanding phases — all consist essentially in the invention of the kind of tools described above, i.e. procedures guided by explicit rules come to replace certain more or less instinctive procedures in the activity of thinking and especially

the activity of deductive inference. It is important to realize that this development did not reach its end in the construction of the syntactical method. Some essential features in the contemporary work of logicians are guided by instinct and common sense, although they could be guided by explicit rules. These rules, however, would be not syntactical but semantical. This will become clear if we give a few examples from contemporary logical investigations.

One of the important questions investigated in modern logic is that of the completeness of given logical systems. Sometimes this question is meant in a clearly syntactical way; it is the question whether a given calculus is such that every sentence belonging to it is either provable or refutable (i.e. its negation is provable). In other cases, the question of completeness is meant in another sense. Take for example Gödel's theorem of 1930 concerning the completeness of a certain calculus (the so-called lower functional calculus similar to FC, but containing predicate variables). He formulates it in the following two ways: (1) "Every formula (i.e. sentential function of the calculus in question) which is universally valid is provable"; (2) "Every formula is either refutable or satisfiable." We find two different kinds of terms occurring here. The terms 'provable' and 'refutable' are obviously syntactical. They are exactly defined on the basis of the rules of the calculus in question; and those rules are explicitly stated in the form of primitive sentences (axioms) and rules of inference. Thus we are given everything required for an exact understanding and use of these terms. Not so for the terms 'universally valid' ("allgemeingültig") and 'satisfiable' ("erfüllbar"). They are explained in the following way: a formula (a sentential function of the calculus in question) is called universally valid if it is true for all values of the free variables; it is called satisfiable if there are values of the free variables for which it is true. Clearly these two terms are not of a syntactical but of a semantical nature. In a theory of semantics they could be exactly defined on the basis of the concept of enti-

ties satisfying a sentential function (this is the basic concept in Tarski's semantics, see 'fulfillment', Volume I, § 11). Gödel's theorem is accordingly of a peculiar nature which is usually not recognized: it combines syntactical and semantical concepts; in a more exact formulation it would state a relation between a syntactical and a corresponding semantical system. The terms 'universally valid' and 'satisfiable' play an important role in contemporary logical investigations, especially in problems of completeness and in the so-called decision-problem ("Entscheidungsproblem"). Other terms of a semantical nature which are frequently used are 'true', 'false', 'truth-value', 'values of a variable', etc. The decisive point is this: while the syntactical terms used by logicians are exactly defined and belong to a well-constructed and recognized theory (namely syntax), the same is not true for the semantical terms. These are merely explained in an informal manner, without a theory as framework for them. No rules constituting semantical systems corresponding to the calculi in question are given; although such rules would serve as a basis for the semantical terms used. Thus the understanding and the use of these terms is left to common-sense and instinct. It is assumed that the reader knows how to interpret and use them on the basis of his knowledge of everyday language. This assumption is perhaps correct to some extent. Similarly, however, most people know how to use the terms 'all' and 'some' before a logician expounds Aristotle's rules to them. Once we concede that it is essential for the development of logic to give explicit rules for all terms which play a central role, then we see that the demand for such rules in the case of the semantical terms is at least as urgent as in the case of 'all' and 'some'. It should be noted that the semantical terms used in recent investigations do not merely serve for incidental explanations or illustrations outside of the theory dealt with, but are essential to that theory; this is shown by the fact that they occur in the very formulations of the problems and the theorems.

It should be clear that the foregoing remarks concerning customary formulations in contemporary logic are not meant as a criticism of the authors, but simply as a critical description of the present status of the metalanguage commonly used by logicians. I wish merely to call attention to the fact that this customary language contains both syntactical and semantical terms. Once we are aware of this fact, we can see that, in order to improve the method of logic, we need a systematically constructed semantics as urgently as we previously needed a systematically constructed syntax (theory of proof). [Hilbert and Bernays, in *Grundlagen der Mathematik*, vol. I, distinguish between two theories, called theory of proof ("Beweistheorie") and set-theoretic logic ("mengentheoretische Logik") respectively. From their explanations it becomes clear that, in our terminology, the first is syntax, the second is semantics. The explanations given for set-theoretic logic may indeed be regarded as the beginning of a systematization of semantics. The fundamental difference between those discussions in the book mentioned which are syntactical and those which are semantical would become clearer if the distinction between expressions and their designata were observed more strictly.]

The value of semantics for philosophy and science

In the course of these last few decades the importance of logical analysis — sometimes called analysis of language, sometimes analysis of knowledge — for theoretical philosophy and for the methodology of science has been more and more widely acknowledged. Many of us even hold the view — first emphasized by Russell, and substantiated by his work — that logic is the very foundation of philosophical and methodological investigation. Hence, if it is true that the progress of logic in its present phase requires the development of a systematic semantics, the indirect value of semantics for philosophy and science becomes clear. It is the purpose of these Studies to help in the construction of se-

mantics — that was the special aim of the first volume — and then to show possibilities of its application. The present volume gives an example of an application to a fundamental problem in logic; an application to philosophical problems in the narrower sense is not here intended. Some very brief indications of the relevance of semantics for certain philosophical problems have been given in the appendix to Volume I (§ 38). That scientists in talking about theories and hypotheses continually use concepts which belong to L-semantics, has been shown by a few examples in Volume I, pp. 61 f. A few first steps, still rather elementary and tentative, towards an application of semantics to the methodology of empirical science have been made in my Encyclopedia monograph (see Bibliography). I am convinced that many other workers will soon recognize the value of semantics as an instrument of logical analysis, will help in developing and improving this instrument, and will then apply it to the clarification and solution of their special problems in various fields.

The next volume

In the next volume of these Studies in Semantics, I intend to deal with modal logic, i.e. the theory of such concepts as logical necessity, possibility, impossibility, etc. (see Volume I, § 38d). It is amazing that modal logic, having been originated in its modern form by C. I. Lewis in 1918, has not made any essential progress since then. There have been numerous publications in this field, some of them, especially in recent years, with interesting and fruitful results. However, all these investigations continue to confine themselves to the same field as Lewis' systems: they investigate the modalities in connection with the most elementary logical system, namely propositional logic. It seems that as yet the modalities have not been introduced into the more important logic of functions. The construction of this more interesting but also much more complex system, both in semantical and in syntactical form, will be the chief task of the next volume. Then, in

addition to logical modalities, other kinds of modalities will be studied, among them the concepts of causal necessity, possibility, etc. Further, the question will be discussed whether modal concepts (in the widest sense, including all concepts which are not extensional or truth-functional, compare § 12) are useful or even necessary in certain special fields, e.g. in the metalanguage used for semantics and perhaps in psychology, in statements concerning believing, knowing, and similar propositional attitudes.

Acknowledgments

The first draft of the manuscript for this book, containing the chief results (the possibility of non-normal interpretations for PC, and a full formalization with the help of junctives), was written in the autumn of 1938. I wish to express my gratitude to the University of Chicago for releasing me from teaching duties during that quarter. (What is now Volume I was written later when it became clear during the writing of the present book that a separate, systematic explanation of the semantical concepts used was necessary.) As in the case of the first volume, I again am indebted to the Department of Philosophy at Harvard University and to the American Council of Learned Societies for grants in aid of publication. I want to thank Dr. Abraham Kaplan, who again helped me in the preparation of the manuscript.

R. C.

SANTE FE, NOVEMBER 1942.

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FORMALIZATION OF LOGIC

§ 1. Introduction: The Problem of a Full Formalization of Logic

The problem is whether — and in what way — it is possible to construct a calculus as a full formalization of propositional and functional logic, i.e. such that the principal logical signs can be interpreted only in the normal way. New basic concepts for syntax and semantics will be required for this purpose.

In Volume I of these Studies in Semantics, we have developed concepts of syntax referring to calculi and concepts of semantics referring to semantical systems. Further, there were concepts relating semantical systems and calculi, especially the concepts of the different kinds of interpretations — true, false, L-true, L-false interpretations, etc.

If we look at a calculus from the point of view of semantics, then we might say that it formalizes certain semantical features of expressions. Thus e.g. the fact that a certain sentence \mathfrak{S}_1 is true is itself of a semantical, not a syntactical, nature. But it can be formalized, i.e. mirrored in a syntactical way, if a calculus K is constructed in such a way that \mathfrak{S}_1 is C-true in K . Analogously, the equivalence of \mathfrak{S}_2 and \mathfrak{S}_3 may be mirrored by their C-equivalence in K . But L-concepts also may be mirrored formally, e.g. L-truth by C-truth, L-implication by C-implication. In general, we might define the concept of a formalization of a semantical property in the following way. A radical semantical property F of an expression \mathfrak{A}_i is *formalized* in $K =_{Df} \mathfrak{A}_i$ has the property F in every semantical system which is a true interpretation for K . And an L-semantical property F of \mathfrak{A}_i is formalized in $K =_{Df} \mathfrak{A}_i$ has F in every L-true interpretation for K . Analogously for semantical relations.

Having a certain designatum is also a semantical property

of an expression. It is easy to see that, in the case of a descriptive sign, a property of this kind cannot in general be formalized. Thus e.g. it is not possible to formalize the property of 'a' designating Chicago and the property of 'P' designating the property of being large — in other words, it is not possible to construct a calculus K in such a way that in every true interpretation for K 'a' and 'P' have the designata mentioned. If a true interpretation for K with these designata is given, another true interpretation for K with different designata can always be constructed.

Whether or not logic can be completely formalized is an important question for the foundations of logic. If the question is taken simply in the ordinary sense, as referring to a formalization of logical deduction — in other words, to a formalization of the relation of L-implication — then the answer is of course in the affirmative. L-implication can in general be formally represented by C-implication (concerning some difficulties and qualifications, see [Foundations] § 10, at the end). But we will take the question here in a stronger sense. If a calculus K containing the ordinary connectives of propositional logic could be constructed in such a way that it would formalize all essential properties of these connectives so that it would exclude the possibility of interpreting the connectives in any other than the ordinary way, then we should say that K was a *full formalization of propositional logic*. And if K should, in addition, impose the ordinary interpretation on the universal and existential operators, we should speak of a *full formalization of junctional logic*. The principal problem to be dealt with in this book is the question whether, and how, a full formalization of logic is possible, in the sense just indicated, which will be made more precise later. It is well known — it was shown first by E. L. Post — that the concept of L-truth within propositional logic is formalized in the ordinary propositional

calculus, which we call PC; the same holds for L-implication. Further, it is easy to see that some essential logical properties of the connectives reveal themselves in the L-truth of certain sentences and in the L-implication between certain sentences in which the connectives occur. [Thus, e.g., it is characteristic of the sign of negation ' \sim ' and the sign of disjunction ' \vee ' that every sentence is an L-implicate of $\{\mathcal{S}_i, \sim \mathcal{S}_i\}$, that $\mathcal{S}_i \vee \mathcal{S}_j$ is an L-implicate of \mathcal{S}_i and also of \mathcal{S}_j , that $\mathcal{S}_i \vee \sim \mathcal{S}_i$ is L-true, etc.] Thus one might perhaps be led to the assumption that PC is a full formalization for propositional logic. The subsequent discussions (Chapter C), however, will come to the surprising conclusion that this is not the case. We shall find non-normal interpretations for PC — that is to say, true and even L-true interpretations for PC in which the connectives have an interpretation different from the normal one as given by the normal truth-tables (NTT). And this holds not only for PC but likewise for any other calculus constructed with the help of the customary syntactical concepts. In spite of this, a full formalization will be found to be possible by the construction of a new calculus PC* (Chapter E). This, however, requires entirely new basic concepts for syntax. These concepts will be applicable not only to the propositional calculus but to calculi in general, and likewise to semantical systems (Chapter D).

The investigation of propositional logic will take up the greater part of this book. The results can then easily be extended so as to apply to functional logic (Chapter F). The result is analogous. The ordinary functional calculus FC (taken here with individual variables for a denumerable field of individuals, without predicate variables) admits of non-normal interpretations for the universal and existential operators, as is well known. A new calculus FC* will be constructed on the basis of PC* such that it imposes the normal interpretation upon the operators.

At several places in [I] (i.e. Volume I of these Studies; see Bibliography) we found symptoms of a thoroughgoing lack of symmetry in the foundations of semantics and syntax (e.g. in [I] pp. 38f., 72, 77, and 172). We shall find that the introduction of the new concepts will remove these defects and thereby lead to a simpler and more uniform structure of the system of concepts both in syntax and in semantics.

The development toward a formalization of logic begins, in a certain way, with the very beginning of systematic logic, in Aristotle. Leibniz emphasized the formal method in his construction of various calculi. But his ideas were all but forgotten by his successors, until a new development began about the middle of the last century with the creation of symbolic logic. It was Frege (1893) above all who recognized the importance of the formal method and carried it through in an exact way, while simultaneously insisting that a logical system should not be regarded merely as a formal calculus but should, in addition, be understood as expressing thoughts.

It is to be noted that we use the term '*formal*' here always in the strict sense of "in abstraction from the meaning", hence as synonymous with 'syntactical' (see [I] § 37, 'Formal', meaning III, and [I] p. 10), in contradistinction to the weaker meanings: "general" (meaning I), and "logically valid" (meaning II). The difference between II and III might be described in this way: in using the term 'formal' in meaning II, abstraction is made from the meaning of the descriptive signs but not from that of the logical signs. [Thus, for instance, the sentence ' $P(a) \vee \sim P(a)$ ' is called *formally* true (II) because its truth is logically necessary on the basis of the meaning of ' \vee ' and ' \sim ' (as given by the truth-tables), independent of the meaning of ' P ' and ' a '.] On the other hand, in the method which we call formal (in meaning III) or syntactical, abstraction is made from the meaning of all signs, including the logical ones. [For instance, in a suitable calculus, the sentence ' $P(a) \vee \sim P(a)$ ' is shown to be C-true (provable) on the basis of rules which are formal in the strict sense III inasmuch as they do not refer to the meaning of any signs, not even of the connectives.]

A. THE PROPOSITIONAL CALCULUS (PC)

Chapter A contains an analysis of the ordinary propositional calculus PC. Different forms of PC are distinguished. The four singular extensional connectives (e.g. negation) and the sixteen binary (e.g. disjunction) are syntactically characterized. Syntactical theorems concerning the connectives in PC are proved. This chapter serves chiefly to prepare for the later discussions in Chapters B and C.

§ 2. The Calculus PC_1

PC_1 is the Hilbert-Bernays form of PC, with signs of negation and disjunction as the only connectives.

In what follows, we shall use the C-terminology for syntax ([I] § 28). The following table shows the correspondence between the customary terms and the C-terms.

CUSTOMARY TERMS	C-TERMS
derivable	C-implicate
directly derivable	direct C-implicate
provable	C-true
primitive sentence	direct C-implicate of Λ
refutable	C-false
directly refutable	directly C-false
equipollent	C-equivalent
decidable	C-determinate
undecidable	C-indeterminate

The correspondence of terms in the table above is here, with respect to PC, a strict synonymy. Since PC does not contain a rule of refutation, 'refutable in PC' and 'C-false in PC' are both empty, 'C-implicate in PC' coincides with 'derivable in PC', and 'C-true in PC' coincides with 'provable in PC' ([I] T29-54). Sometimes, but not frequently, a rule of refutation has been added to PC. It seems that in every calculus of this kind which has been constructed so far, every directly C-false (directly refutable) \mathfrak{A} , is such that every sentence is

derivable from it. Therefore, for these calculi as well, the coincidences mentioned hold ([I] T29-55).

We shall use 'PC' as a common name for the different forms of the ordinary propositional calculus. (We shall later explain more in detail which calculi are meant as forms of PC.) The different forms vary with respect to the choice of primitive signs, primitive sentences, and rules of inference, but they are known to agree with respect to possible results of proofs and derivations. Hence, if two forms contain the same sentences, they are coincident, although not directly coincident, calculi ([I] D₃₁₋₉ and 8).

As an example of a form of PC, we shall take here the one constructed by Hilbert and Bernays (it is constructed out of Russell's form in [Princ. Math.] by a simplification due to Bernays). It uses as primitive signs those of negation_C ('~') and disjunction_C ('∨') (the subscript 'C' will be explained in § 3). We call this form of PC the calculus PC₁. Another similar form will be called PC₁^D; it contains further connectives defined on the basis of the two primitives mentioned. PC₁ does not contain rules of refutation.

A general connective (D₁) is a sign that can be applied to any closed sentences as components (arguments). A connective is said to be of **degree** n if it is applied to n components. Connectives of degree one are also called **singular** connectives, those of degree two **binary**. (As in [I], the more important definitions and theorems are marked by '+'.)

+D2-1. a_i is a **general connective** of degree n in a calculus K (or in a semantical system S) =_{D₁} K (or S) contains closed sentences, and for every n -term sequence of closed sentences in K (or S respectively) there is a full sentence of a_i in K (or S) with that sequence of components.

If a_k is a singular and a_l a binary general connective, then

we designate the full sentence of a_k with the component \mathcal{S}_i by ' $a_k(\mathcal{S}_i)$ ' and the full sentence of a_l with the components \mathcal{S}_i and \mathcal{S}_j by ' $a_l(\mathcal{S}_i, \mathcal{S}_j)$ '.

+D2-2. K contains PC_1 with neg_C as *sign of negation*_C and dis_C as *sign of disjunction*_C =_{Def} the calculus K fulfills the following conditions:

- a. neg_C is a singular and dis_C a binary general connective in K .
- b. The relation of **direct C-implication** (\xrightarrow{dC}) holds in the following cases for any \mathcal{S}_i , \mathcal{S}_j , and \mathcal{S}_k (but not necessarily only in these cases):
 1. $\Lambda \xrightarrow{dC} dis_C(neg_C(dis_C(\mathcal{S}_i, \mathcal{S}_i)), \mathcal{S}_i)$.
 2. $\Lambda \xrightarrow{dC} dis_C(neg_C(\mathcal{S}_i), dis_C(\mathcal{S}_i, \mathcal{S}_j))$.
 3. $\Lambda \xrightarrow{dC} dis_C(neg_C(dis_C(\mathcal{S}_i, \mathcal{S}_j)), dis_C(\mathcal{S}_j, \mathcal{S}_i))$.
 4. $\Lambda \xrightarrow{dC} dis_C(neg_C(dis_C(neg_C(\mathcal{S}_i), \mathcal{S}_j)), dis_C(neg_C(dis_C(\mathcal{S}_k, \mathcal{S}_i)), dis_C(\mathcal{S}_k, \mathcal{S}_j)))$.
 5. $\{\mathcal{S}_i, dis_C(neg_C(\mathcal{S}_i), \mathcal{S}_j)\} \xrightarrow{dC} \mathcal{S}_j$.

By (1) to (4) all sentences of four specified forms are declared to be direct C-implicates of Λ , in other words, primitive sentences in K (see the customary formulation below). (5) is the rule of implication. The definition does not exclude the possibility that K contains still other rules of deduction, e.g. further cases for direct C-implication or rules of refutation.

The *customary formulation* of the rules of deduction for PC_1 with propositional variables is the following:

Primitive sentences of PC_1 :

- a. ' $\sim(p \vee p) \vee p$ '.
- b. ' $\sim p \vee (p \vee q)$ '.
- c. ' $\sim(p \vee q) \vee (q \vee p)$ '.
- d. ' $\sim(\sim p \vee q) \vee (\sim(r \vee p) \vee (r \vee q))$ '.

Rules of inference for PC_1 :

- a. Rule of substitution. From \mathcal{S}_i , $\mathcal{S}_i \left(\frac{I_k}{\mathcal{S}_i} \right)$ is directly derivable.

- b. Rule of implication (in disjunctive form). From \mathcal{S}_i and $\sim \mathcal{S}_i \vee \mathcal{S}_j$, \mathcal{S}_j is directly derivable.

D₂ is formulated in such a way — as is often done — that no propositional variables are required; such variables may or may not occur in K . But K cannot have only propositional variables as ultimate components for the connectives; it must also contain closed sentences (see D₁). In D₂, not merely four sentences but an infinite number of sentences are taken as primitive sentences (D₂b, 1 to 4); these are the same sentences as those which, in the form just mentioned, are constructed out of the primitive sentences by any substitutions. Hence, in D₂, no rule of substitution is necessary.

An example of a calculus containing PC₁ is K₁, described in [I] §§ 27 and 30.

§ 3. Propositional Connections_C in PC

Syntactical concepts for the four singular and the sixteen binary propositional connections_C are introduced (see table). PC₁^D is a calculus containing primitive signs of negation_C and disjunction_C and defined signs for the other connections_C.

We shall summarize in this section some of the known features of the propositional connections occurring in PC. There are two customary ways of constructing a system for the propositional connectives, one by the use of primitive sentences and rules of inference, the other by the use of truth-tables. The second method, however, gives truth-conditions for the sentences and thereby interprets them. Hence, it does not belong to syntax but to semantics. Therefore the name 'Propositional Calculus' is appropriate only to a system of the first kind. For a system of truth-tables, that term, although customarily used, might better be replaced by a term like 'Propositional Logic'.

There is also a syntactical method analogous to that of the truth-tables. It uses tables with arbitrary values (e.g. numerical values) or unspecified values instead of truth-values. This method, in contradistinction to that of truth-tables proper, can also be used with any

other number of values than two (so-called many-valued systems). Instead of the term 'truth-table', the wider term 'value-table', which does not prejudge the question of the interpretation of the values, should be used; often the term 'matrix' is used. A system based on formal tables of this kind is then a syntactical system, a calculus. [For explanations of the formal method of value-tables ("method of matrices"), see Łukasiewicz and Tarski [Untersuchungen], pp. 3, 4.]

In propositional logic there are four singular and sixteen binary extensional connectives (see § 10). In a system of PC, corresponding connectives are used. In the interpretation of PC most frequently used (we shall call it the normal interpretation) the connectives are interpreted as the corresponding extensional connectives; this is the reason for their customary names (e.g. 'sign of negation', 'sign of disjunction', etc.) even in syntax. We shall use these names here, but with the subscript 'C' added (see D2-2 and 3 and the subsequent table, column (2)). It is, however, to be noted that we do not intend by this to decide on the interpretation of the connectives. If S is a true or even an L-true interpretation for K , then a sign of negation_C in K is not necessarily a sign of negation in S . ['Sign of negation_C' is a syntactical term to be defined in this section; 'sign of negation' is a semantical term to be defined by D11-23. Sometimes we shall also write 'connections_C' in order to emphasize the syntactical nature of this concept.] In examples, we shall often make use of the customary signs ' \sim ' and ' \vee '; in more exact formulations in the syntax language, however, we make use not of these customary signs, enclosed in quotation marks, but of their syntactical names 'neg_C' and 'dis_C' (see D2-2), thus leaving the particular shapes of the signs undetermined.

The *table* contains syntactical expressions referring to the $4 + 16$ propositional connections_C and connectives, i.e. signs of connections_C, with the exception of the few examples of customary connectives in column (3), which belong, of

course, to the object language. The reason for distinguishing just four singular and sixteen binary connections_c is of a semantical rather than a syntactical nature; it will become clear in the later explanation of the semantical concepts of the corresponding connections (§ 10).

SYNTACTICAL CONCEPTS OF PROPOSITIONAL CONNECTIONS_c AND
CONNECTIVES IN PC

(1)	(2)	(3)	(4)	(5)
Connections _c		Connectives		
Abbr- viation	Ordinary Name	Custom- ary Symbol	Syntac- tical Name	Expression in PC ₁

I. The four singular connections_c

cConn ₁ ¹	tautology _c		cb ₁	$\mathcal{E}, V \sim \mathcal{E}_i$
cConn ₂ ¹	(identity) _c		cb ₂	\mathcal{E}_i
cConn ₃ ¹	negation _c	\sim	cb ₃ (neg _c)	$\sim \mathcal{E}_i$
cConn ₄ ¹	contradiction _c		cb ₄	$\sim (\mathcal{E}_i V \sim \mathcal{E}_i)$

II. The sixteen binary connections_c

cConn ₁ ²	tautology _c		cc ₁	$\mathcal{E}, V \sim \mathcal{E}_i$
cConn ₂ ²	disjunction _c	\vee	cc ₂ (disc)	$\mathcal{E}, V \mathcal{E}_i$
cConn ₃ ²	(inverse implication) _c		cc ₃	$\mathcal{E}_i V \sim \mathcal{E}_i$
cConn ₄ ²	(first component)		cc ₄	\mathcal{E}_i
cConn ₅ ²	implication _c	\supset	cc ₅ (imp _c)	$\sim \mathcal{E}_i V \mathcal{E}_i$
cConn ₆ ²	(second component)		cc ₆	\mathcal{E}_i
cConn ₇ ²	equivalence _c	$=$	cc ₇ (equ _c)	$\sim (\sim \mathcal{E}_i V \sim \mathcal{E}_i) V \sim (\mathcal{E}_i V \mathcal{E}_i)$
cConn ₈ ²	conjunction _c	\cdot	cc ₈ (conc)	$\sim (\sim \mathcal{E}_i V \sim \mathcal{E}_i)$
cConn ₉ ²	exclusion _c	$ $	cc ₉	$\sim \mathcal{E}_i V \sim \mathcal{E}_i$
cConn ₁₀ ²	(non-equivalence) _c		cc ₁₀	$\sim (\sim \mathcal{E}_i V \mathcal{E}_i) V \sim (\mathcal{E}_i V \sim \mathcal{E}_i)$
cConn ₁₁ ²	(negation _c of second)		cc ₁₁	$\sim \mathcal{E}_i$
cConn ₁₂ ²	(first alone)		cc ₁₂	$\sim (\sim \mathcal{E}_i V \mathcal{E}_i)$
cConn ₁₃ ²	(negation _c of first)		cc ₁₃	$\sim \mathcal{E}_i$
cConn ₁₄ ²	(second alone)		cc ₁₄	$\sim (\mathcal{E}_i V \sim \mathcal{E}_i)$
cConn ₁₅ ²	bi-negation _c		cc ₁₅	$\sim (\mathcal{E}_i V \mathcal{E}_i)$
cConn ₁₆ ²	contradiction _c		cc ₁₆	$\sim (\mathcal{E}_i V \sim \mathcal{E}_i)$

The terms 'sign of negation_C in PC₁' and 'sign of disjunction_C in PC₁' have been defined in D2-2; 'neg_C' and 'dis_C' are used as names of signs of these kinds. We shall now define syntactical terms for the other connections_C, listed in columns (1) and (2) of the table with respect to PC₁ (D1 and 2; see the later example for D2(5)). The more general concepts with respect to any other form of PC will be defined in § 4.

+D3-1. (1) \mathfrak{S}_k is a *sentence of ${}_C\text{Conn}_1^1$* (or a *tautology_C sentence*) with \mathfrak{S}_i (as component) in PC₁ in $K =_{\text{Df}} K$ contains PC₁ and \mathfrak{S}_k is C-equivalent in K to $\text{dis}_C(\mathfrak{S}_i, \text{neg}_C(\mathfrak{S}_i))$.

(3) \mathfrak{S}_k is a *sentence of ${}_C\text{Conn}_3^1$* (or a *negation_C sentence*) with \mathfrak{S}_i (as component) in PC₁ in $K =_{\text{Df}} K$ contains PC₁ and \mathfrak{S}_k is C-equivalent in K to $\text{neg}_C(\mathfrak{S}_i)$.

(2) and (4) are analogous; see explanation below.

+D3-2. (1) \mathfrak{S}_k is a *sentence of ${}_C\text{Conn}_1^2$* (or a *tautology_C sentence*) with \mathfrak{S}_i and \mathfrak{S}_j (as components) in PC₁ in $K =_{\text{Df}} K$ contains PC₁ and \mathfrak{S}_k is C-equivalent in K to $\text{dis}_C(\mathfrak{S}_i, \text{neg}_C(\mathfrak{S}_i))$.

(2) \mathfrak{S}_k is a *sentence of ${}_C\text{Conn}_2^2$* (or a *disjunction_C sentence*) with \mathfrak{S}_i and \mathfrak{S}_j in PC₁ in $K =_{\text{Df}} K$ contains PC₁ and \mathfrak{S}_k is C-equivalent in K to $\text{dis}_C(\mathfrak{S}_i, \mathfrak{S}_j)$.

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(5) \mathfrak{S}_k is a *sentence of ${}_C\text{Conn}_5^2$* (or an *implication_C sentence*) with \mathfrak{S}_i and \mathfrak{S}_j in PC₁ in $K =_{\text{Df}} K$ contains PC₁ and \mathfrak{S}_k is C-equivalent in K to $\text{dis}_C(\text{neg}_C(\mathfrak{S}_i), \mathfrak{S}_j)$.

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(8) \mathfrak{S}_k is a *sentence of ${}_C\text{Conn}_8^2$* (or a *conjunction_C sentence*) with \mathfrak{S}_i and \mathfrak{S}_j in PC₁ in $K =_{\text{Df}} K$ contains PC₁ and \mathfrak{S}_k is C-equivalent in K to $\text{neg}_C(\text{dis}_C(\text{neg}_C(\mathfrak{S}_i), \text{neg}_C(\mathfrak{S}_j)))$.

(3), (4), (6), (7), (9) to (16) are analogous. Each of the four definitions in D₁ and the sixteen in D₂ is constructed in the following way. The terms in the definiendum are those in columns (1) and (2) of the table; the sentence mentioned at the end of the definiens is that described in column (5) of the table on the line in question (where, for the sake of brevity, ' $\sim \mathcal{S}_m$ ' is written for ' $\text{neg}_C(\mathcal{S}_m)$ ', and ' $\mathcal{S}_m \vee \mathcal{S}_n$ ' for ' $\text{dis}_C(\mathcal{S}_m, \mathcal{S}_n)$ ').

D₃ is a definition schema furnishing four definitions if the numerals '1' to '4' are taken as subscripts in the place of 'q'. Likewise, D₄ furnishes sixteen definitions with '1' to '16' in the place of 'r'. Subsequent definitions, theorems, and explanations containing a subscript variable 'q' or 'r' are to be understood analogously.

+D3-3. α_k is a *sign* (or *connective*) for ${}_C\text{Conn}_q^1$ ($q = 1$ to 4) in PC₁ in $K =_{\text{Df}} K$ contains PC₁, α_k is a general connective in K , and, for any closed sentence \mathcal{S}_i in K , the full sentence $\alpha_k(\mathcal{S}_i)$ is a sentence for ${}_C\text{Conn}_q^1$ in PC₁ in K .

+D3-4. α_k is a *sign* (or *connective*) for ${}_C\text{Conn}_r^2$ ($r = 1$ to 16) in PC₁ in $K =_{\text{Df}} K$ contains PC₁, α_k is a general connective in K , and, for any closed sentences \mathcal{S}_i and \mathcal{S}_j in K , the full sentence $\alpha_k(\mathcal{S}_i, \mathcal{S}_j)$ is a sentence for ${}_C\text{Conn}_r^2$ in PC₁ in K .

If there is a sign for ${}_C\text{Conn}_q^1$ ($q = 1$ to 4) in PC₁ in K , we shall use ' ${}_Cb_q$ ' as a syntactical name for it; analogously, ' ${}_Cc_r$ ' ($r = 1$ to 16) for a sign for ${}_C\text{Conn}_r^2$ (column (4) of the table). Instead of ' ${}_Cb_3$ ' we usually write '**neg_C**'. Instead of ' ${}_Cc_2$ ' we usually write '**dis_C**'; likewise, for $r = 5, 7$, or 8 , we usually use '**imp_C**', '**equ_C**', and '**con_C**' respectively.

Examples, for $r = 5$. The sentence $\text{dis}_C(\text{neg}_C(\mathcal{S}_i), \mathcal{S}_j)$, and likewise any other sentence which is C-equivalent to it, is called a sentence of ${}_C\text{Conn}_5^2$ or a sentence of implication_C with \mathcal{S}_i and \mathcal{S}_j in PC₁ in K (D₂(5)). Thus, if K contains PC₁, it always contains implication_C sentences, even if the signs of negation_C and disjunction_C are the only connectives. If K contains a general connective α_k , such that

its full sentence with any closed components \mathcal{S}_i and \mathcal{S}_i is always C-equivalent to the sentence $\text{disc}(\text{neg}_C(\mathcal{S}_i), \mathcal{S}_i)$, then a_k is called a sign of cConn_C^2 (D4(5)) or of implication_C, and 'ct_s' or 'imp_C' is used as a name for it.

The expressions in column (5) of the table show how all singular and binary connections_C can be expressed in PC₁. Therefore, these expressions may be taken as definientia in definitions of signs for these connections_C, on the basis of the signs for negation_C and disjunction_C as primitives.

Example. A definition of a sign of conjunction_C ct_s may be formulated as follows: "ct_s($\mathcal{S}_1, \mathcal{S}_2$) for $\text{neg}_C(\text{disc}(\text{neg}_C(\mathcal{S}_1), \text{neg}_C(\mathcal{S}_2)))$ ". Compare [I] § 24 concerning definition sentences and definition rules. A definition rule is here regarded as an additional rule of inference, which states that two sentences (e.g. ' $A \supset C \cdot D$ ' and ' $A \supset \sim(\sim C \vee \sim D)$ ' in the above example) which differ only in two expressions of the forms of the definiendum (' $C \cdot D$ ') and the definiens (' $\sim(\sim C \vee \sim D)$ ') are direct C-implicates of each other.

The form of PC containing all the definitions indicated by column (5) on the basis of PC₁ will be called PC₁^D (D6). Hence, PC₁^D contains a connective for each of the 4 + 16 connections_C listed.

+D3-6. K contains PC₁^D =_{df} K contains PC₁ and, in addition, definition rules on the basis of the signs of negation_C and disjunction_C for signs for all other singular and binary connections_C, with definientia as given in column (5) of the above table.

'a [b]' in D7, and analogously in some of the subsequent definitions, theorems, and proofs, means that (a), i.e. here D3-7a, is to be read without the *insertions in square brackets*, while for (b), here D3-7b, these insertions are to be added (or sometimes to be taken instead of the preceding expression).

D3-7a [b]. \mathfrak{I}_j is a C-implicate of \mathfrak{I}_i in K by PC₁ [PC₁^D] =_{df} K contains PC₁ [PC₁^D] and $\mathfrak{I}_i \rightarrow \mathfrak{I}_j$ in virtue of the rules

of deduction as given in D2-2b [and, in addition, the definition rules of PC_1^D as described in D3-6]. Analogously for any other C-term defined on the basis of 'C-implication'.

§ 4. Forms of PC

The general concept of a calculus containing any form of PC and the concepts of the propositional connections_C in a calculus of this kind are defined.

Under what conditions shall we say that K contains a form of PC? K need not contain all the $4 + 16$ connectives of PC_1^D ; it would suffice if K contained e.g. signs of negation_C and disjunction_C (the primitives in PC_1) or signs of negation_C and conjunction_C (the primitives in another form, PC_2 ; see below). Suppose that K_n is a sub-calculus of a calculus K_m containing PC_1^D , and that some of the connectives in PC_1^D occur in K_n . Then under suitable conditions we shall say that K_n contains a form of PC. First, we shall require that, if $\mathfrak{X}_i \rightarrow \mathfrak{X}_j$ in K_m and \mathfrak{X}_i and \mathfrak{X}_j belong to K_n too, then $\mathfrak{X}_i \rightarrow \mathfrak{X}_j$ in K_n ; in other words, that K_n is a conservative sub-calculus of K_m ([I] D31-7; PC_1^D usually does not contain rules of refutation). Second, K_n must not be too poor a sub-calculus; if it contained e.g. a sign of conjunction_C as the only connective we should not say that it contained a form of PC. K_n must contain a sufficient set of connectives for building sentences for all $4 + 16$ connections_C. This can be formulated in a syntactical way by requiring that K_n be a sub-calculus of K_m containing for every sentence \mathfrak{S}_i in K_m a C-equivalent sentence. (Thus, e.g., if K_n contains PC_2 , this requirement is fulfilled, because for any sentence containing any connectives of PC_1^D there is a C-equivalent sentence with the connectives of negation_C and conjunction_C only.) But we have to admit still other calculi. Suppose that the connectives used in K_p happen to be different from

those in K_n but in such a way that they correspond strictly to those in K_n ; in other words, that K_p is isomorphic to K_n (see [I] D31-10). In this case also we should say that K_p contained a form of PC. These considerations lead to D1. It is to be noticed that K_m and K_n may be identical; likewise K_n and K_p , and hence K_m and K_p .

D4-1. A calculus K_p **contains** (a form of) **PC** = **D1**, there are calculi K_m and K_n such that the following conditions are fulfilled:

- a. K_m contains PC_1^D ;
- b. K_n is a conservative sub-calculus of K_m ;
- c. for every sentence \mathfrak{S}_i in K_m there is a sentence \mathfrak{S}_j in K_n (and K_m) which is C-equivalent to \mathfrak{S}_i in K_m ;
- d. K_p is isomorphic to K_n by a correlation H .

The following definition is analogous to D3-7.

D4-2. \mathfrak{X}_j is a **C-implicate** of \mathfrak{X}_i in K by **PC** = **D1** K contains PC by being isomorphic by a correlation H with a sub-calculus of a calculus K_m containing PC_1^D , and $\mathfrak{X}'_i \vec{C} \mathfrak{X}'_j$ in K_m by PC_1^D , where \mathfrak{X}'_i is the correlate in K_m of \mathfrak{X}_i by H and \mathfrak{X}'_j that of \mathfrak{X}_j . Analogously for any other C-term defined by 'C-implication'.

Now we can easily define the syntactical concepts 'sign of negation_C', etc., with respect to any form of PC.

D4-3. a_i is a *sign (connective)* for $cConn_q^1$ ($q = 1$ to 4) or $cConn_r^2$ ($r = 1$ to 16) in PC in $K = D1$ K contains PC by being isomorphic by a correlation H to a sub-calculus of a calculus K_m containing PC_1^D , and a_i is correlated by H to a sign for the same connection_C (i.e. $cConn_q^1$ or $cConn_r^2$, respectively) in PC_1 in K_m .

As previously in PC_1 , now in general in PC, we shall designate a sign for $cConn_q^1$ by $c b_q$ and a sign for $cConn_r^2$ by $c c_r$.

A number of other forms of PC besides PC_1 are known. Thus e.g. each of the following sets of primitive signs is a sufficient basis for expressing all connections_C: signs for negation_C and conjunction_C (PC_2); negation_C and implication_C (PC_3); exclusion_C (PC_4 , shown by Sheffer), bi-negation_C (PC_5 , Sheffer). Suitable rules of deduction for these forms have been constructed for PC_3 by Frege, for PC_4 by Nicod and Quine; for PC_5 by Quine.

The systems mentioned are only a few examples. For each of the bases mentioned, there is an infinite number of different forms. Further, there are other bases besides those mentioned. For instance, a sign of negation_C together with cc_3 or cc_{12} or cc_{14} yields a sufficient basis; each of these systems is similar to PC_3 and to PC_2 since implication_C and conjunction_C can easily be expressed or defined. [Definientia for $\text{imp}_C(\mathcal{S}_i, \mathcal{S}_j)$: $cc_3(\mathcal{S}_i, \mathcal{S}_j)$, $\text{neg}_C(cc_{12}(\mathcal{S}_i, \mathcal{S}_j))$, and $\text{neg}_C(cc_{14}(\mathcal{S}_i, \mathcal{S}_j))$ respectively.]

The following sections (§§ 5 to 9) contain theorems concerning not PC in isolation but, rather, calculi containing PC. This difference seems slight, but it is essential for the later discussion of interpretation. Sometimes calculi are constructed in symbolic logic which do not contain PC as a part, but, so to speak, represent *PC* itself in a pure form, i.e. as a calculus containing propositional variables as the only ultimate components (see "the customary formulation", at the end of § 2). But in a calculus of this kind, every sentence is open and is either C-true or C-comprehensive (i.e. every sentence is a C-implicate of it, [1] D30-6). This is a disadvantage for a discussion of interpretations. The customary interpretation is L-true, and hence all sentences in a pure form of PC become here L-determinate; there are no factual sentences. Moreover, the most convenient and customary formulation of semantical rules for the normal interpretation, namely the truth-tables, cannot be directly used for such a form of PC, because the truth-tables apply only to closed sentences (see remarks on the rules of NTT, § 10). Therefore, for the discussion of interpretations we shall have to take into consideration not pure

forms but calculi containing PC with or without propositional variables, but in any case containing closed sentences and hence other constants in addition to the connectives (in the simplest case propositional constants). For this reason, the syntactical discussions in the following sections will likewise refer not to a pure form of PC but to calculi containing PC with closed sentences. It is true that this will make the theorems more complicated. It turns out that the results also depend to some extent upon certain features of the calculus *K* in question outside of PC, e.g. upon the additional rules of deduction of *K*. But it seems that by means of this the results formulated in the theorems of this chapter are more generally applicable. In the practical work in symbolic logic and in the logical foundations of mathematics, we deal in most cases not with pure forms of PC but with calculi containing PC. Therefore it is important to see in what respect the features of the more comprehensive calculus influence the properties of the propositional connections that occur.

§ 5. Elementary Theorems Concerning PC

Some well-known elementary syntactical theorems concerning the propositional connections in PC are listed for later reference.

Before we come to the discussion of our chief problem, namely the normal interpretation of PC (§§ 10 and 11) and the question of the possibility of non-normal interpretations (beginning in § 15), we must study the syntactical features of PC, independent of any interpretation. This is the task of the rest of this chapter (§§ 5 to 9). The present section lists only some elementary and well-known theorems for the convenience of later reference. These theorems state some examples of C-truth, C-implication, and C-equivalence.

Proofs are not given because they are either known or easily constructed with the help of those known. [Derivations for the cases listed in T2 and 3 may be found by first constructing a proof for the corresponding implication_C sentences with the help of the conjunctive normal form; see e.g. Hilbert [Logik], Kap. I, §§ 3 and 4.] It is essential that the theorems refer not only to the forms of PC but to any calculus *K* containing such a form; the theorems hold for any sentences of *K* no matter what other signs besides those of PC they may contain. This is especially important for theorems like T2l.

Here, and in the further discussions as well, for the sake of simplicity, we shall refer mostly to the special form PC₁. But, as can easily be seen on the basis of D4-1, the results hold likewise for any other sentences which are C-equivalent to those mentioned here by PC₁ or PC₁^D or any other form with different primitive signs but the same connectives; and they hold also for the correlated sentences in any other form of PC. [Thus, for instance, if something is said about $\text{dis}_C(\text{neg}_C(\mathfrak{S}_i), \mathfrak{S}_j)$, i.e. $\sim \mathfrak{S}_i \vee \mathfrak{S}_j$, in PC₁, then the same holds for $\mathfrak{S}_i \supset \mathfrak{S}_j$, for $\sim(\mathfrak{S}_i \cdot \sim \mathfrak{S}_j)$, etc., in PC₁^D, and for any corresponding sentences in any other form of PC.]

T5-1. If *K* contains PC₁, then any sentence of one of the following forms is *C-true* in *K* by PC₁:

- a. $\text{dis}_C(\mathfrak{S}_i, \text{neg}_C(\mathfrak{S}_i))$.
- b. $\text{dis}_C(\text{neg}_C(\mathfrak{S}_i), \mathfrak{S}_i)$.

T5-2. If *K* contains PC₁ or (for (h) and (q) to (t)) PC₁^D, then in each of the following cases \mathfrak{S}_i is a *C-implicate* of \mathfrak{T}_i in *K* by PC₁ or PC₁^D, respectively.

\mathcal{I}_i is:	\mathcal{E}_j is:
a. $\text{disc}(\mathcal{E}_m, \mathcal{E}_m)$	\mathcal{E}_m
b. \mathcal{E}_m	$\text{disc}(\mathcal{E}_m, \mathcal{E}_n)$
c. \mathcal{E}_n	$\text{disc}(\mathcal{E}_m, \mathcal{E}_n)$
e. $\{\text{disc}(\mathcal{E}_m, \mathcal{E}_n), \text{negc}(\mathcal{E}_n)\}$	\mathcal{E}_n
f. $\{\text{disc}(\mathcal{E}_m, \mathcal{E}_n), \text{negc}(\mathcal{E}_n)\}$	\mathcal{E}_m
g. $\{\text{disc}(\mathcal{E}_m, \text{negc}(\mathcal{E}_n)), \mathcal{E}_n\}$	\mathcal{E}_m
h. $\{\text{disc}(\mathcal{E}_m, \mathcal{E}_n), \text{impc}(\mathcal{E}_m, \mathcal{E}_n)\}$	\mathcal{E}_n
i. $\{\text{disc}(\mathcal{E}_m, \mathcal{E}_n), \text{disc}(\mathcal{E}_m, \text{negc}(\mathcal{E}_n))\}$	\mathcal{E}_m
k. $\{\text{disc}(\mathcal{E}_k, \text{disc}(\text{negc}(\mathcal{E}_m), \mathcal{E}_n)), \text{disc}(\mathcal{E}_k, \mathcal{E}_m)\}$	$\text{disc}(\mathcal{E}_k, \mathcal{E}_n)$
l. $\{\mathcal{E}_m, \text{negc}(\mathcal{E}_m)\}$	any sentence (in K)
m. $\text{negc}(\text{disc}(\text{negc}(\mathcal{E}_m), \text{negc}(\mathcal{E}_n)))$	\mathcal{E}_m
n. $\text{negc}(\text{disc}(\text{negc}(\mathcal{E}_m), \text{negc}(\mathcal{E}_n)))$	\mathcal{E}_n
q. $\text{concc}(\mathcal{E}_m, \mathcal{E}_n)$	\mathcal{E}_m
r. $\text{concc}(\mathcal{E}_m, \mathcal{E}_n)$	\mathcal{E}_n
t. $\{\text{disc}(\mathcal{E}_k, \text{impc}(\mathcal{E}_m, \mathcal{E}_n)), \text{disc}(\mathcal{E}_k, \mathcal{E}_m)\}$	$\text{disc}(\mathcal{E}_k, \mathcal{E}_n)$

T5-3. If K contains PC_1 or (for (n) to (u)) PC_1^D , then in each of the following cases \mathcal{I}_i and \mathcal{I}_j are *C-equivalent* in K by PC_1 or PC_1^D , respectively.

\mathcal{I}_i is:	\mathcal{I}_j is:
a. \mathcal{E}_m	$\text{negc}(\text{negc}(\mathcal{E}_m))$
b. $\{\mathcal{E}_m, \mathcal{E}_n\}$	$\text{negc}(\text{disc}(\text{negc}(\mathcal{E}_m), \text{negc}(\mathcal{E}_n)))$
d. $\text{disc}(\mathcal{E}_m, \mathcal{E}_n)$	$\text{disc}(\mathcal{E}_n, \mathcal{E}_m)$
i. $\{\text{disc}(\text{negc}(\mathcal{E}_m), \mathcal{E}_i), \text{disc}(\text{negc}(\mathcal{E}_n), \mathcal{E}_i)\}$	$\text{disc}(\text{negc}(\text{disc}(\mathcal{E}_m, \mathcal{E}_n)), \mathcal{E}_i)$
j. $\text{disc}(\mathcal{E}_m, \text{disc}(\mathcal{E}_n, \mathcal{E}_p))$	$\text{disc}(\mathcal{E}_n, \text{disc}(\mathcal{E}_m, \mathcal{E}_p))$
k. $\text{disc}(\text{disc}(\mathcal{E}_m, \mathcal{E}_n), \mathcal{E}_p)$	$\text{disc}(\mathcal{E}_m, \text{disc}(\mathcal{E}_n, \mathcal{E}_p))$
l. $\text{disc}(\mathcal{E}_m, \mathcal{E}_m)$	\mathcal{E}_m
n. $\text{concc}(\mathcal{E}_m, \mathcal{E}_n)$	$\text{negc}(\text{disc}(\text{negc}(\mathcal{E}_m), \text{negc}(\mathcal{E}_n)))$
p. $\{\mathcal{E}_m, \mathcal{E}_n\}$	$\text{concc}(\mathcal{E}_m, \mathcal{E}_n)$
r. $\text{concc}(\mathcal{E}_m, \mathcal{E}_n)$	$\text{concc}(\mathcal{E}_n, \mathcal{E}_m)$
s. $\text{impc}(\mathcal{E}_m, \mathcal{E}_n)$	$\text{disc}(\text{negc}(\mathcal{E}_m), \mathcal{E}_n)$
t. $\text{disc}(\mathcal{E}_m, \mathcal{E}_n)$	$\text{impc}(\text{negc}(\mathcal{E}_m), \mathcal{E}_n)$
u. $\{\text{impc}(\mathcal{E}_m, \mathcal{E}_i), \text{impc}(\mathcal{E}_n, \mathcal{E}_i)\}$	$\text{impc}(\text{disc}(\mathcal{E}_m, \mathcal{E}_n), \mathcal{E}_i)$

§ 6. Extensible Rules

The concept 'extensible rule of inference' is defined. In itself it is not important, but it is needed for some later theorems. In a first reading, *this section may be left out*.

If a calculus K contains PC and, in addition, other rules of deduction, then it is not so much the additional primitive sentences as the additional rules of inference which have an influence upon the syntactical properties of the propositional connections_c in K . In the present section, the property of extensibility which a rule of inference in a calculus containing PC may or may not have will be defined and studied. The results will be used in the later discussion.

Let us regard as an example the rule of substitution for propositional variables (i), as it often occurs in calculi containing PC. According to it, $\mathfrak{S}_j(\mathfrak{S}_n^{(i)})$ is a direct C-implicate of \mathfrak{S}_j in K . If now we add any closed sentence in K , say \mathfrak{S}_k , as a left-hand disjunctive component to each of those two sentences, we get $\text{dis}_c(\mathfrak{S}_k, \mathfrak{S}_j(\mathfrak{S}_n^{(i)}))$ and $\text{dis}_c(\mathfrak{S}_k, \mathfrak{S}_j)$. It can easily be seen that for these two sentences the relation of C-implication in K still holds. The same holds generally for any application of the rule of substitution even if \mathfrak{S}_k is not closed, provided only that \mathfrak{S}_k does not contain a free variable which occurs freely in \mathfrak{S}_j or \mathfrak{S}_n (see below, proof for T3a). We shall formulate this result by saying that the rule mentioned is *extensible* (with respect to a left-hand disjunctive component).

We shall now define this concept in a general way. We take ' $\text{dis}'_c(\mathfrak{S}_k, \mathfrak{R}_i)$ ' as designation of that sentential class which we construct out of \mathfrak{R}_i by adding \mathfrak{S}_k as left-hand disjunctive component to every sentence of \mathfrak{R}_i (thus e.g. transforming $\{\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3, \dots\}$ into $\{\mathfrak{S}_k \vee \mathfrak{S}_1, \mathfrak{S}_k \vee \mathfrak{S}_2, \mathfrak{S}_k \vee \mathfrak{S}_3, \dots\}$). The sentences of $\text{dis}''_c(\mathfrak{R}_i, \mathfrak{S}_k)$ are constructed out of

those of \mathfrak{R}_i by adding \mathfrak{S}_k as right-hand disjunctive component. Analogous designations are formed for other connections_C (and likewise for the normal connectives in a semantical system; see § 10).

D6-1. A rule of inference R in a calculus K is (extendible with respect to a disjunctive component, or briefly) **extendible** =_{DI} K contains PC; and for any \mathfrak{S}_j , \mathfrak{R}_i , and \mathfrak{S}_k in K , if \mathfrak{S}_j is a direct C-implicate of \mathfrak{R}_i in virtue of R , and \mathfrak{S}_k is either closed (i.e. does not contain a free variable) or at least does not contain a free variable also occurring freely in \mathfrak{S}_j or any sentence of \mathfrak{R}_i , then $\text{dis}'_C(\mathfrak{S}_k, \mathfrak{R}_i) \xrightarrow{C} \text{dis}_C(\mathfrak{S}_k, \mathfrak{S}_j)$ in K .

In an analogous way we may define 'extendible with respect to a left-hand implicative component' by the condition that $\text{imp}'_C(\mathfrak{S}_k, \mathfrak{R}_i) \xrightarrow{C} \text{imp}_C(\mathfrak{S}_k, \mathfrak{S}_j)$, and 'extendible with respect to a conjunctive component' by the condition that $\text{con}'_C(\mathfrak{S}_k, \mathfrak{R}_i) \xrightarrow{C} \text{con}_C(\mathfrak{S}_k, \mathfrak{S}_j)$. But these terms will be used only here in T1 and 2. [In the case of disjunction_C and conjunction_C we need not distinguish between extendibility with respect to left-hand and with respect to right-hand component, because these connections_C are commutative (T5-3d and r).]

The reason for the restricting condition with respect to \mathfrak{S}_k in D1 will be explained later (see remarks on T28-10).

T6-1. If a rule is extendible with respect to a disjunctive component, then it is also extendible with respect to a left-hand implicative component; and vice versa. (From T5-3s and t.)

We shall see later that certain theorems in general syntax hold only for those calculi whose rules are extendible with respect to a disjunctive component; and this is the reason for introducing this concept. But there is no need in any

theorem for an analogous condition with respect to a conjunctive component, because every rule fulfills this condition (T2).

T6-2. If K contains PC, then any rule of inference of K (and hence also any instance of C-implication in K with a non-empty premiss class) is extensible with respect to a conjunctive component.

Proof. Let \mathcal{S}_i be a direct C-implicate or a C-implicate of a non-empty class \mathcal{R}_i in K , and \mathcal{S}_k any sentence in K . For every sentence \mathcal{S}_i of \mathcal{R}_i , \mathcal{S}_i and \mathcal{S}_k are C-implicates of $\text{conc}(\mathcal{S}_k, \mathcal{S}_i)$ (T5-2q and r) and hence of $\text{conc}(\mathcal{S}_k, \mathcal{R}_i)$ ([I] T29-33 and 44). Hence each of the following items is a C-implicate of $\text{conc}(\mathcal{S}_k, \mathcal{R}_i)$: a. \mathcal{R}_i ([I] T29-40); b. $\{\mathcal{S}_k, \mathcal{S}_i\}$ (the same); c. $\text{conc}(\mathcal{S}_k, \mathcal{S}_i)$ ((b), T5-3p).

Many forms of PC contain the rule of implication (or separation or abruption): " \mathcal{S}_j is a direct C-implicate of \mathcal{S}_i and the implication_C sentence of \mathcal{S}_i and \mathcal{S}_j ". In some forms the implication_C sentence is formed with the help of a sign of implication_C; in other forms, as e.g. PC₁, it is formed as $\text{disc}(\text{neg}_C(\mathcal{S}_i), \mathcal{S}_j)$. Thus we distinguish rules of implication in implicative and in disjunctive form.

T6-3. In any calculus containing PC, each of the following rules, if it occurs, is extensible: a. the rule of substitution for propositional variables; b. the rule of implication in implicative form; c. the rule of implication in disjunctive form.

Proof. a. Suppose that \mathcal{S}_k does not contain f_m as a free variable. Then $\text{disc}(\mathcal{S}_k, \mathcal{S}_i(\mathcal{S}_n^m))$ is the same as $(\text{disc}(\mathcal{S}_k, \mathcal{S}_i))(\mathcal{S}_n^m)$ and, hence, is a C-implicate of $\text{disc}(\mathcal{S}_k, \mathcal{S}_i)$. — b. From T5-2t. — c. From T5-2k.

T6-4. In any calculus containing PC, any definition rule (of the customary form; see § 3) is extensible.

Proof. If an application of a definition rule leads from \mathcal{S}_i to \mathcal{S}_j , then, for any \mathcal{S}_k , an application of the same rule leads from $\text{disc}(\mathcal{S}_k, \mathcal{S}_i)$ to $\text{disc}(\mathcal{S}_k, \mathcal{S}_j)$.

T6-5. If K contains PC₁ (either in the form given in § 2

or in a form using propositional variables and a rule of substitution) or PC_1^D (§ 3), then all rules of inference of PC_1 or PC_1^D in K are extensible. (From T3a, c, and T4.)

It can moreover be shown that, if K contains any form of PC, then all rules of inference of PC in K are extensible. It will be shown later that the rules of inference in the Lower Functional Calculus FC_1 are also extensible (T28-10).

As remarked previously, the subsequent theorems in § 6 to 9 refer to PC_1 or PC_1^D , but they hold likewise for any other form of PC.

T6-10. Let K fulfill the following three conditions:

- A. K contains PC_1 ,
- B. all rules of inference in K are extensible,
- C. K either contains no rule of refutation or, if it does, every directly C-false \mathfrak{T}_i in K is such that every sentence in K is derivable from it.

For any non-empty \mathfrak{R}_i , \mathfrak{S}_j , and \mathfrak{S}_k such that \mathfrak{S}_k does not contain any free variable occurring freely in \mathfrak{S}_j or in any sentence of \mathfrak{R}_i , if $\mathfrak{R}_i \vdash \mathfrak{S}_j$ in K , then $\text{dis}_C'(\mathfrak{S}_k, \mathfrak{R}_i) \vdash \text{dis}_C(\mathfrak{S}_k, \mathfrak{S}_j)$.

Proof. Because of (C), 'C-implicate in K ' and 'derivable in K ' coincide ([I] T29-54a and 55a). Hence, if $\mathfrak{R}_i \vdash \mathfrak{S}_j$, then there is a derivation D_1 leading from \mathfrak{R}_i to \mathfrak{S}_j . We transform D_1 into the sequence of sentences D_2 by adding \mathfrak{S}_k as a left-hand disjunctive component to every sentence. (D_2 is not necessarily itself a derivation but is the skeleton of a derivation D_3 which leads from $\text{dis}_C'(\mathfrak{S}_k, \mathfrak{R}_i)$ to $\text{dis}_C(\mathfrak{S}_k, \mathfrak{S}_j)$.) Every sentence \mathfrak{S}_i in D_1 is either (a) a sentence of \mathfrak{R}_i or (b) a primitive sentence of K or (c) a direct C-implicate of a class \mathfrak{R}_p of preceding sentences in virtue of a rule of inference. In the case (a), $\text{dis}_C(\mathfrak{S}_k, \mathfrak{S}_i)$ is a sentence of $\text{dis}_C'(\mathfrak{S}_k, \mathfrak{R}_i)$. In the case (b), $\text{dis}_C(\mathfrak{S}_k, \mathfrak{S}_i)$ is C-true in K (T5-2c). The first sentence of D_2 belongs either to (a) or to (b); therefore (a): it is a C-implicate of $\text{dis}_C'(\mathfrak{S}_k, \mathfrak{R}_i)$ ([I] T29-33 and 74). In the case (c), because of condition (B), $\text{dis}_C(\mathfrak{S}_k, \mathfrak{S}_i)$ is a C-implicate of $\text{dis}_C'(\mathfrak{S}_k, \mathfrak{R}_i)$, which is a class of preceding sentences in D_2 . Thus (β): for any sentence \mathfrak{S}_m in D_2 the following holds: if every sentence preceding \mathfrak{S}_m is a C-implicate of $\text{dis}_C'(\mathfrak{S}_k, \mathfrak{R}_i)$, then the

same holds for \mathfrak{S}_m ([I] T29-40 and 44). According to the principle of induction (transfinite induction if D_1 is a transfinite derivation; see [I] § 25 at the end), it follows from (a) and (β) that every sentence of D_2 , and hence also $\text{dis}_C(\mathfrak{S}_k, \mathfrak{S}_i)$, is a C-implicate of $\text{dis}_C'(\mathfrak{S}_k, \mathfrak{R}_i)$.

+**T6-12.** Let K fulfill the conditions (A), (B), and (C) in T10. If \mathfrak{S}_i is closed and $\mathfrak{S}_i \xrightarrow{C} \mathfrak{S}_j$ in K , then any implication_C sentence with \mathfrak{S}_i and \mathfrak{S}_j , e.g. $\text{dis}_C(\text{neg}_C(\mathfrak{S}_i), \mathfrak{S}_j)$, is C-true in K .

Proof. Under the conditions stated, $\text{dis}_C(\text{neg}_C(\mathfrak{S}_i), \mathfrak{S}_i)$ is C-true in K (T5-1b). Further, $\text{dis}_C(\text{neg}_C(\mathfrak{S}_i), \mathfrak{S}_i) \xrightarrow{C} \text{dis}_C(\text{neg}_C(\mathfrak{S}_i), \mathfrak{S}_j)$ in K , because $\mathfrak{S}_i \xrightarrow{C} \mathfrak{S}_j$ and $\text{neg}_C(\mathfrak{S}_i)$ is closed (T10). Therefore $\text{dis}_C(\text{neg}_C(\mathfrak{S}_i), \mathfrak{S}_j)$ is also C-true in K ([I] T29-70). Any other implication_C sentence is C-equivalent to this sentence (D3-2 (5)) and hence also C-true.

The reason for the condition in T12 that \mathfrak{S}_i be closed becomes clear by the following counter-examples. 1. In a calculus containing propositional variables, ' q ' is a (direct) C-implicate of ' p ' (by substitution), but ' $\sim p \vee q$ ' is not C-true. 2. In the functional calculus (§ 28), ' $P(x)$ ' \xrightarrow{C} ' $P(a)$ ', but ' $\sim P(x) \vee P(a)$ ' (or ' $P(x) \supset P(a)$ '), which is C-equivalent to ' $(x)(P(x) \supset P(a))$ ', is not C-true.

In T12, \mathfrak{S}_i is required to be closed. It would not suffice to require that \mathfrak{S}_i do not contain a free variable which does not occur as a free variable in \mathfrak{S}_j . This is shown by the following counter-example. ' p ' \xrightarrow{C} ' $\sim p$ ' (by substitution); but ' $\sim p \vee \sim p$ ', which is C-equivalent to ' $\sim p$ ', is not C-true.

T12 is a theorem of general syntax. It may be called the *general deduction theorem*. T14b and T28-11 are special applications of this theorem for the propositional and the functional calculi respectively. A theorem similar to T28-11 has been called deduction theorem by Hilbert and Bernays [Grundl. Math. I] p. 155.

T6-14. If K contains PC_1 or PC_1^D and there are no other rules of inference in K than those of PC_1 or PC_1^D , then the following holds:

- a. For any non-empty \mathfrak{R}_i , \mathfrak{S}_j , and \mathfrak{S}_k such that \mathfrak{S}_k does not contain any free variable occurring freely in \mathfrak{S}_j or in any sentence of \mathfrak{R}_i , if $\mathfrak{R}_i \xrightarrow{C} \mathfrak{S}_j$,

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in K , then $\text{dis}'_C(\mathcal{S}_k, \mathfrak{R}_i) \xrightarrow{C} \text{dis}_C(\mathcal{S}_k, \mathcal{S}_j)$. (From T5 and 10.)

+b. If \mathcal{S}_i is closed and $\mathcal{S}_i \xrightarrow{C} \mathcal{S}_j$, then any implication_C sentence with \mathcal{S}_i and \mathcal{S}_j is C-true in K . (From T5 and 12.)

The same holds also for the functional calculus (T28-11).

If, in constructing a calculus, one finds that some rule of inference is not extensible, there is reason to doubt whether the calculus fulfills the purpose for which it was intended. It is then easy to transform the calculus in the following way into a stronger one whose rules are extensible. A rule R saying " $\mathfrak{R}_i \xrightarrow{\text{dC}} \mathcal{S}_j$ if such and such conditions are fulfilled" is replaced by a rule R' : " $\text{dis}'_C(\mathcal{S}_i, \mathfrak{R}_i) \xrightarrow{\text{dC}} \text{dis}_C(\mathcal{S}_i, \mathcal{S}_j)$ if \mathcal{S}_i does not contain a free variable occurring freely in \mathcal{S}_j , or in any sentence of \mathfrak{R}_i , and if such and such other conditions are fulfilled".

It is easy to see that any rule of the form R' is extensible. If \mathcal{S}_k is any sentence which does not contain a free variable occurring freely in \mathcal{S}_i or in \mathcal{S}_j or in any sentence of \mathfrak{R}_i , then $\text{dis}'_C(\text{dis}_C(\mathcal{S}_k, \mathcal{S}_i), \mathfrak{R}_i) \xrightarrow{\text{dC}} \text{dis}_C(\text{dis}_C(\mathcal{S}_i, \mathcal{S}_i), \mathcal{S}_j)$, according to R' (taking $\text{dis}_C(\mathcal{S}_k, \mathcal{S}_i)$ in the place of \mathcal{S}_i). Therefore, according to the associative law for disjunction (T5-3k), $\text{dis}'_C(\mathcal{S}_k, \text{dis}'_C(\mathcal{S}_i, \mathfrak{R}_i)) \xrightarrow{C} \text{dis}_C(\mathcal{S}_k, \text{dis}_C(\mathcal{S}_i, \mathcal{S}_j))$.

For an example of the transformation of R into R' see remarks on T28-10 concerning rule (11').

§ 7. General Theorems Concerning Disjunction_C

Some general syntactical theorems concerning disjunction_C are proved. One of the results: under ordinary conditions, two signs of disjunction_C are C-interchangeable (T4b).

The following theorems are proved with respect to the signs of disjunction_C and negation_C in PC₁. According to the previous discussion, they hold likewise for any other

form of PC with respect to the connections_c of ${}_c\text{Conn}_2^2$ (i.e. disjunction_c) and ${}_c\text{Conn}_3^1$ (i.e. negation_c), no matter whether there are connectives for these connections_c or not.

T7-1. If K contains PC_1 and any implication_c sentence with \mathcal{S}_m and \mathcal{S}_n , e.g. $\text{dis}_c(\text{neg}_c(\mathcal{S}_m), \mathcal{S}_n)$, is C-true in K , then $\mathcal{S}_m \rightarrow_c \mathcal{S}_n$ in K . (From D3-2(5), D2-2b(5), [I] T29-81.)

+T7-2. Let K fulfill the conditions (A), (B), and (C) in T6-10. Then the following holds:

- a. $\text{dis}_c(\mathcal{S}_i, \mathcal{S}_j)$ is a C-implicate in K both of \mathcal{S}_i and of \mathcal{S}_j . (From T5-2b, c.)
- b. If \mathcal{S}_i and \mathcal{S}_j are closed, $\text{dis}_c(\mathcal{S}_i, \mathcal{S}_j)$ is a strongest sentence in K with the property (a); that is to say, if any \mathcal{S}_l is a C-implicate both of \mathcal{S}_i and of \mathcal{S}_j , then $\text{dis}_c(\mathcal{S}_i, \mathcal{S}_j) \rightarrow_c \mathcal{S}_l$.

Proof for (b). If the conditions mentioned are fulfilled, both $\text{dis}_c(\text{neg}_c(\mathcal{S}_i), \mathcal{S}_l)$ and $\text{dis}_c(\text{neg}_c(\mathcal{S}_j), \mathcal{S}_l)$ are C-true (T6-12). Hence the class of these two sentences is C-true ([I] T29-72), and likewise $\text{dis}_c(\text{neg}_c(\text{dis}_c(\mathcal{S}_i, \mathcal{S}_j)), \mathcal{S}_l)$, because it is a C-implicate of that class (T5-3i, [I] T29-70). Therefore $\text{dis}_c(\mathcal{S}_i, \mathcal{S}_j) \rightarrow_c \mathcal{S}_l$ (T1).

T7-3. Let K fulfill the conditions (A), (B), and (C) in T6-10. Let \mathcal{S}_i , \mathcal{S}'_i , \mathcal{S}_j , and \mathcal{S}'_j be any closed sentences in K such that \mathcal{S}_i is C-equivalent to \mathcal{S}'_i and likewise \mathcal{S}_j to \mathcal{S}'_j . Then $\text{dis}_c(\mathcal{S}_i, \mathcal{S}_j)$ is C-equivalent to $\text{dis}_c(\mathcal{S}'_i, \mathcal{S}'_j)$.

Proof. Since \mathcal{S}_i is C-equivalent to \mathcal{S}'_i , $\text{dis}_c(\mathcal{S}_i, \mathcal{S}_j)$ is C-equivalent to $\text{dis}_c(\mathcal{S}'_i, \mathcal{S}_j)$ (T6-10), to $\text{dis}_c(\mathcal{S}'_i, \mathcal{S}_i)$ (T5-3d), and further, because of the C-equivalence of \mathcal{S}_i and \mathcal{S}'_i , to $\text{dis}_c(\mathcal{S}'_i, \mathcal{S}'_i)$ (T6-10) and to $\text{dis}_c(\mathcal{S}'_i, \mathcal{S}'_j)$ (T5-3d).

Condition (A) in T4, below, refers to a calculus K containing PC_1 twice with two signs of disjunction_c. This is meant in the following way. K contains two sets of rules of deduction as required in D2-2. The signs of negation_c may or may not be identical. If the first set of rules (four primitive sentences and the rule of implication) refers, say, to ' \sim ' and

' \vee ', then the second refers either to ' \sim ' and ' \vee ' or to ' \sim ' and ' \vee '. In any case, according to D2-2a, the three or four connectives are general connectives (D2-1), and hence connectives of the two sets may occur within one sentence.

Condition (D) in T₄ is fulfilled also by most of the non-extensional (intensional) calculi constructed so far, e.g. by Lewis' calculus of Strict Implication and the numerous similar calculi by other authors.

+T7-4. Conditions for K :

A. K contains PC₁ twice, with two different signs of disjunction_C, dis_{C1} and dis_{C2}.

B and C, as in T6-10.

D. K is either extensional in relation to partial sentences ([I] D₃₁₋₁₈) or, if not, every non-extensional primitive connective in K , say a_i , fulfills the following condition: if every two corresponding arguments in two full sentences \mathcal{S}_m and \mathcal{S}_n of a_i are C-equivalent, then \mathcal{S}_m and \mathcal{S}_n are C-equivalent.

a. If K fulfills the conditions (A), (B), and (C), then for any closed \mathcal{S}_i and \mathcal{S}_j , dis_{C1}($\mathcal{S}_i, \mathcal{S}_j$) and dis_{C2}($\mathcal{S}_i, \mathcal{S}_j$) are C-equivalent in K .

b. If K fulfills the four conditions (A), (B), (C), and (D), then dis_{C1} and dis_{C2} are C-interchangeable in K ([I] D₃₁₋₁₃).

Proof. a. Let \mathcal{S}_i and \mathcal{S}_j be any closed sentences. Then dis_{C1}($\mathcal{S}_i, \mathcal{S}_j$) is a C-implicate both of \mathcal{S}_i and of \mathcal{S}_j (T2a); and likewise dis_{C2}($\mathcal{S}_i, \mathcal{S}_j$). Therefore, the second disjunction_C sentence is a C-implicate of the first (T2b), and the first of the second. Hence they are C-equivalent.

b. Since the two disjunction_C sentences mentioned are C-equivalent (a), they are C-interchangeable (condition (D), [I] T₃₁₋₁₀₀). But their mutual replacement in a larger sentence is the same as a mutual replacement of dis_{C1} and dis_{C2}. Therefore these two signs are C-interchangeable.

We shall come back to T₄ in a later discussion of possible

interpretations for signs of disjunction_C. T4 holds also for other forms of PC than PC₁, and even for two different forms.

§ 8. General Theorems Concerning Negation_C

Some general syntactical theorems concerning negation_C are proved. One of the results: under ordinary conditions, two signs of negation_C are C-interchangeable (T9b).

+T8-1. Let K fulfill the conditions (A), (B), and (C) in T6-10. Let \mathcal{S}_i be any closed sentence in K .

- a. Every sentence in K which is a C-implicate both of \mathcal{S}_i and of $\text{neg}_C(\mathcal{S}_i)$ is C-true.
- b. $\text{neg}_C(\mathcal{S}_i)$ is a strongest sentence which has the relation to \mathcal{S}_i stated in (a), that is to say, if \mathcal{S}_l is such that every sentence which is a C-implicate both of \mathcal{S}_i and of \mathcal{S}_l is C-true, then $\text{neg}_C(\mathcal{S}_i) \xrightarrow{C} \mathcal{S}_l$.

Proof. a. Let \mathcal{S}_j be a C-implicate both of \mathcal{S}_i and of $\text{neg}_C(\mathcal{S}_i)$. Then $\text{disc}(\mathcal{S}_i, \text{neg}_C(\mathcal{S}_i)) \xrightarrow{C} \mathcal{S}_j$ (T7-2b). Therefore, since $\text{disc}(\mathcal{S}_i, \text{neg}_C(\mathcal{S}_i))$ is C-true (T5-1a), \mathcal{S}_j is C-true ([I] T29-70).

b. Let \mathcal{S}_l fulfill the conditions stated. Then $\text{disc}(\mathcal{S}_i, \mathcal{S}_l)$, being a C-implicate both of \mathcal{S}_i and of \mathcal{S}_l (T7-2a), must be C-true. $\text{disc}(\text{neg}_C(\mathcal{S}_i), \mathcal{S}_l)$ is C-equivalent to $\text{disc}(\mathcal{S}_i, \mathcal{S}_l)$ (T5-3a, T7-3) and hence is likewise C-true. Therefore $\text{neg}_C(\mathcal{S}_i) \xrightarrow{C} \mathcal{S}_l$ (T7-1).

T8-2. Let K contain PC₁.

- a. If \mathcal{S}_i is C-true in K , $\text{neg}_C(\mathcal{S}_i)$ is C-comprehensive. ([I] D30-6).
- b. If $\text{neg}_C(\mathcal{S}_i)$ is C-true, \mathcal{S}_i is C-comprehensive.

Proof. a. Let \mathcal{S}_i be C-true. Every sentence in K is a C-implicate of $\{\mathcal{S}_i, \text{neg}_C(\mathcal{S}_i)\}$ (T5-21) and hence a C-implicate of $\text{neg}_C(\mathcal{S}_i)$ ([I] T29-81). Therefore, $\text{neg}_C(\mathcal{S}_i)$ is C-comprehensive ([I] D30-6). — b. If $\text{neg}_C(\mathcal{S}_i)$ is C-true, $\text{neg}_C(\text{neg}_C(\mathcal{S}_i))$ is C-comprehensive (a), and hence \mathcal{S}_i also (T5-3a, [I] T30-49).

It is to be noted that T₂, unlike T₃, does not impose restricting conditions on K and \mathcal{S}_i .

T8-3. Let K fulfill the conditions (A), (B), and (C) in T6-10. Let \mathcal{S}_i be any closed sentence in K .

- a. If \mathcal{S}_i is C-comprehensive, $\text{neg}_C(\mathcal{S}_i)$ is C-true.
- b. If $\text{neg}_C(\mathcal{S}_i)$ is C-comprehensive, \mathcal{S}_i is C-true.

Proof. a. Let \mathcal{S}_i be C-comprehensive. Then $\mathcal{S}_i \rightarrow_C \text{neg}_C(\mathcal{S}_i)$ ([I] D30-6). Further, $\text{neg}_C(\mathcal{S}_i) \rightarrow_C \text{neg}_C(\mathcal{S}_i)$ ([I] T29-32). Hence, $\text{neg}_C(\mathcal{S}_i)$ is C-true (T1a). — b. Let $\text{neg}_C(\mathcal{S}_i)$ be C-comprehensive. Then $\text{neg}_C(\text{neg}_C(\mathcal{S}_i))$ is C-true (a), and hence also \mathcal{S}_i (T5-3a, [I] T29-70).

T₄, below, is in a certain sense a counterpart to T₁.

T8-4. Let K contain PC₁.

- a. Every sentence in K which C-implies both \mathcal{S}_i and $\text{neg}_C(\mathcal{S}_i)$ is C-comprehensive.
- b. Let K , moreover, fulfill the conditions (B) and (C) in T6-10. Let \mathcal{S}_i be any closed sentence in K . Then $\text{neg}_C(\mathcal{S}_i)$ is a weakest sentence which has the relation to \mathcal{S}_i stated in (a); that is to say, if \mathcal{S}_l is such that every sentence which C-implies both \mathcal{S}_i and \mathcal{S}_l is C-comprehensive, then $\mathcal{S}_l \rightarrow_C \text{neg}_C(\mathcal{S}_i)$.

Proof. a. If $\mathcal{S}_i \rightarrow_C \mathcal{S}_i$ and $\mathcal{S}_i \rightarrow_C \text{neg}_C(\mathcal{S}_i)$, then $\mathcal{S}_i \rightarrow_C \{\mathcal{S}_i, \text{neg}_C(\mathcal{S}_i)\}$ and $\mathcal{S}_i \rightarrow_C$ every sentence (T5-21). — b. Let \mathcal{S}_m be $\text{neg}_C(\text{disc}(\text{neg}_C(\mathcal{S}_i), \text{neg}_C(\mathcal{S}_i)))$, which is a conjunction_C sentence with \mathcal{S}_i and \mathcal{S}_i (D3-2 (8)). Then \mathcal{S}_m C-implies both \mathcal{S}_i and \mathcal{S}_i (T5-2m, n). Let \mathcal{S}_l fulfill the condition stated in the theorem. Then \mathcal{S}_m is C-comprehensive. Therefore, $\text{disc}(\text{neg}_C(\mathcal{S}_i), \text{neg}_C(\mathcal{S}_i))$ is C-true (T3b). Hence, $\mathcal{S}_l \rightarrow_C \text{neg}_C(\mathcal{S}_i)$ (T7-1).

T8-6. Let K fulfill the conditions (A), (B), and (C) in T6-10.

- a. If \mathcal{S}_i is closed and $\mathcal{S}_i \rightarrow_C \mathcal{S}_j$, then $\text{neg}_C(\mathcal{S}_j) \rightarrow_C \text{neg}_C(\mathcal{S}_i)$.

A. THE PROPOSITIONAL CALCULUS (PC)

- b. If \mathcal{S}_i is closed and $\text{neg}_C(\mathcal{S}_i) \not\vdash_C \text{neg}_C(\mathcal{S}_i)$, then
 $\mathcal{S}_i \not\vdash_C \mathcal{S}_j$.

Proof. a. Under the conditions stated, $\text{disc}(\text{neg}_C(\mathcal{S}_i), \mathcal{S}_i)$ is C-true (T6-12). Hence $\text{neg}_C(\mathcal{S}_i) \not\vdash_C \text{neg}_C(\mathcal{S}_i)$ (T5-2f, [I] T29-81). — b. Under the conditions stated, $\text{neg}_C(\text{neg}_C(\mathcal{S}_i)) \not\vdash_C \text{neg}_C(\text{neg}_C(\mathcal{S}_i))$ (a). Therefore, since $\mathcal{S}_i \not\vdash_C \text{neg}_C(\text{neg}_C(\mathcal{S}_i))$ and $\text{neg}_C(\text{neg}_C(\mathcal{S}_i)) \not\vdash_C \mathcal{S}_i$ (T5-3a), $\mathcal{S}_i \not\vdash_C \mathcal{S}_j$ ([I] T29-44).

T8-7. Let K fulfill the conditions (A), (B), and (C) in T6-10. Let \mathcal{S}_i and \mathcal{S}_j be closed and C-equivalent. Then $\text{neg}_C(\mathcal{S}_i)$ and $\text{neg}_C(\mathcal{S}_j)$ are C-equivalent. (From T6a.)

T8-8. Let K fulfill the conditions (A), (B), and (C) in T6-10. Let \mathcal{S}_i and \mathcal{S}_j be constructed in the same way with the help of the signs of negation_C and disjunction_C but out of different components, those of \mathcal{S}_i being $\mathcal{S}_{i1}, \mathcal{S}_{i2}, \dots, \mathcal{S}_{in}$, those of \mathcal{S}_j being $\mathcal{S}_{j1}, \mathcal{S}_{j2}, \dots, \mathcal{S}_{jn}$, all these components being closed. Let any two corresponding components \mathcal{S}_{im} and \mathcal{S}_{jm} ($m = 1$ to n) be C-equivalent in K . Then \mathcal{S}_i and \mathcal{S}_j are C-equivalent. (From T7-3, T7, by inductive inference.)

The following counter-example shows that it is necessary to restrict T7 and 8 to closed components. ' $P(x)$ ' and ' $(x)P(x)$ ' are C-equivalent; but ' $\sim P(x)$ ', which is C-equivalent to ' $(x)(\sim P(x))$ ', is not C-equivalent to ' $\sim(x)P(x)$ '.

The following theorem, T9, requires the occurrence of two signs of negation_C in K . This is to be understood in analogy to the occurrence of two signs of disjunction_C, as explained previously in connection with T7-4.

+**T8-9.** Conditions for K :

A. K contains PC₁ twice with two different signs of negation_C, neg_{C1} and neg_{C2} .

B and C, as in T6-10.

D, as in T7-4.

a. If K fulfills (A), (B), and (C), then, for any

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closed \mathfrak{S}_i , $\text{neg}_{C_1}(\mathfrak{S}_i)$ and $\text{neg}_{C_2}(\mathfrak{S}_i)$ are C-equivalent in K .

- b. If K fulfills (A), (B), (C), and (D), then neg_{C_1} and neg_{C_2} are C-interchangeable in K .

Proof. a. Let \mathfrak{S}_i be closed. Then every sentence which is a C-implicate both of \mathfrak{S}_i and of $\text{neg}_{C_1}(\mathfrak{S}_i)$ is C-true in K (T1a); and every sentence which is a C-implicate both of \mathfrak{S}_i and of $\text{neg}_{C_2}(\mathfrak{S}_i)$ is C-true (T1a). Hence $\text{neg}_{C_2}(\mathfrak{S}_i) \xrightarrow{C} \text{neg}_{C_1}(\mathfrak{S}_i)$ (Trb) and vice versa (Trb). Thus the two sentences are C-equivalent. — b. Proof analogous to that of T7-4b.

§ 9. General Theorems Concerning Other Connections_C

Some syntactical theorems concerning connections_C in general. One of the results: under ordinary conditions, two signs for the same connection_C are C-interchangeable (T4b).

According to the definitions in § 4, any forms of PC correspond in a certain way to PC_1^D and hence to one another. Thus it could easily be shown that if two calculi K_m and K_n contain PC and possess the same or corresponding components, then any syntactical relation like C-implication, C-equivalence, etc., which holds for certain sentences in K_m by PC holds also for the corresponding sentences in K_n by PC. This is true even if K_m and K_n contain different forms of PC; and it is true no matter whether K_m and K_n are entirely separate calculi or are sub-calculi of one calculus K . However, it does not immediately follow from this result that any two corresponding sentences in K_m and K_n as sub-calculi of K are necessarily C-equivalent in K , not even if K_m and K_n contain the same form of PC. That this is the case has been shown above for disjunction_C (T7-4a) and for negation_C (T8-9a). It will now be easy to show the same for the other connections_C in general, because they are expressible in terms of disjunction_C and negation_C.

T9-1. Let K contain PC_1^D and fulfill the conditions (B) and (C) in T6-10. Let \mathcal{S}_i , \mathcal{S}'_i , \mathcal{S}_j , and \mathcal{S}'_j be any closed sentences in K such that \mathcal{S}_i is C-equivalent to \mathcal{S}'_i and \mathcal{S}_j to \mathcal{S}'_j .

- a. For any singularly connective cb_q ($q = 1$ to 4) in PC_1^D in K , $cb_q(\mathcal{S}_i)$ is C-equivalent by PC_1^D in K to $cb_q(\mathcal{S}'_i)$.
- b. For any binary connective cc_r ($r = 1$ to 16) in PC_1^D in K , $cc_r(\mathcal{S}_i, \mathcal{S}_j)$ is C-equivalent by PC_1^D in K to $cc_r(\mathcal{S}'_i, \mathcal{S}'_j)$.

Proof for (b). Let \mathcal{S}_k be the sentence in K formed out of $cc_r(\mathcal{S}_i, \mathcal{S}_j)$ by eliminating the connective cc_r with the help of its definition rule in PC_1^D (D3-6); then \mathcal{S}_k and $cc_r(\mathcal{S}_i, \mathcal{S}_j)$ are C-equivalent by PC_1^D in K . Let \mathcal{S}'_k be formed analogously out of $cc_r(\mathcal{S}'_i, \mathcal{S}'_j)$. Then these two sentences are likewise C-equivalent. \mathcal{S}_k is constructed with the help of signs of negation_C and disjunction_C out of \mathcal{S}_i and \mathcal{S}_j as components, and \mathcal{S}'_k is constructed in the same way out of \mathcal{S}'_i and \mathcal{S}'_j as components. (The common form of \mathcal{S}_k and \mathcal{S}'_k is that given in the table in § 3 in column (5) on the line of $cConn^2$.) Therefore, \mathcal{S}_k is C-equivalent to \mathcal{S}'_k (T8-8). Hence, $cc_r(\mathcal{S}_i, \mathcal{S}_j)$ and $cc_r(\mathcal{S}'_i, \mathcal{S}'_j)$ are likewise C-equivalent. — The proof for (a) is analogous.

T9-2. (Analogous to T8-8.) Let K contain PC_1^D and fulfill the conditions (B) and (C) in T6-10. Let \mathcal{S}_i and \mathcal{S}_j be constructed in the same way with the help of any connectives of PC_1^D but out of different components, those of \mathcal{S}_i being $\mathcal{S}_{i1}, \mathcal{S}_{i2}, \dots, \mathcal{S}_{in}$, those of \mathcal{S}_j being $\mathcal{S}_{j1}, \mathcal{S}_{j2}, \dots, \mathcal{S}_{jn}$, all these components being closed. Let any two corresponding components \mathcal{S}_{im} and \mathcal{S}_{jm} ($m = 1$ to n) be C-equivalent in K . Then \mathcal{S}_i and \mathcal{S}_j are C-equivalent. (From T1, by inductive inference.)

The following theorem, T3, is analogous to T7-4 and T8-9. It refers to two sub-calculi K_m and K_n of K , both containing PC_1^D . This is meant in the same way as explained previously in connection with T7-4. Thus any one of the connectives of PC_1^D in K_m may or may not be identical with the corresponding connective in K_n . The theorem holds likewise if K_m and

K_n contain any other form of PC or even two different forms of PC.

T9-3. Conditions for K :

A. K contains two sub-calculi K_m and K_n , both containing PC_1^D .

B and **C**, as in T6-10.

D, as in T7-4.

Let \mathfrak{S}_m be a sentence constructed with the help of connectives of PC_1^D in K_m out of closed components, and \mathfrak{S}_n be a sentence constructed out of the same components in an analogous way but with the corresponding connectives of PC_1^D in K_n . Then \mathfrak{S}_m and \mathfrak{S}_n are C-equivalent in K .

Proof. Let \mathfrak{S}_m' be the sentence formed out of \mathfrak{S}_m by eliminating all defined connectives of PC_1^D occurring in it. Then \mathfrak{S}_m' is C-equivalent to \mathfrak{S}_m in K_m and hence in K . Let \mathfrak{S}_n' be formed analogously out of \mathfrak{S}_n in K_n . Then \mathfrak{S}_n' is C-equivalent to \mathfrak{S}_n in K_n and in K . \mathfrak{S}_m' and \mathfrak{S}_n' consist of the same components and have analogous forms, but \mathfrak{S}_m' contains the signs of negation_C and disjunction_C of K_m and \mathfrak{S}_n' those of K_n . Now we can transform \mathfrak{S}_m' into \mathfrak{S}_n' by first replacing one occurrence of the sign of disjunction_C in K_m after the other by that in K_n and then doing the same with the signs of negation_C in K_m and K_n . Each step in this transformation leads to a C-equivalent sentence (T7-4, T8-9); therefore \mathfrak{S}_m' and \mathfrak{S}_n' are C-equivalent in K , and hence also \mathfrak{S}_m and \mathfrak{S}_n .

A corollary to T3:

+**T9-4.** Let K fulfill condition (A) in T3 and (B) and (C) in T6-10. Let a_m be a sign of any singulary or binary connection_C in PC_1^D in K_m , and a_n be a sign for the same connection in K_n . Then the following holds:

- a. Two full sentences of a_m and a_n with the same closed component or components are C-equivalent in K .
- b. If K fulfills, moreover, condition (D) in T7-4, then a_m and a_n are C-interchangeable in K .

Proof. (a) is a special case of T3. The proof for (b) is analogous to that for T7-4b.

B. PROPOSITIONAL LOGIC

This chapter deals with propositional logic, i.e. the system of the propositional connections based on the normal truth-tables (NTT). It is a semantical system, in contradistinction to the syntactical system PC. Radical semantical and L-semantical concepts for the connections and for the concept of extensionality are defined.

§ 10. The Normal Truth-Tables (NTT)

The normal truth-tables (NTT) for propositional connections may be regarded as semantical rules stating the truth-conditions for the full sentences of the connectives. A table is given (p. 38) showing the four singulary and the sixteen binary extensional connections with their characteristics, which correspond to the value-columns in the truth-tables.

By the **normal truth-tables** — the system will here be called **NTT** — we mean the customary truth-tables for the singulary and binary propositional connections, regarded as semantical rules (compare [I] § 8). A truth-table for a connection of degree n lists in its first column the 2^n possible distributions of the truth-values T (truth) and F (falsity) among the n components of a full sentence of that connection; in the second column, it gives the truth-value of a full sentence for each of those distributions (see the later examples, truth-tables for negation and disjunction). We restrict ourselves, as is customary, to the singulary and binary connections; all connections of higher degrees can be expressed by negation and disjunction, as Post has shown.

In this way, the truth-table represents a function — we call it the *characteristic function* of the connection — which correlates a truth-value to each of those distributions as arguments. Thus, for instance, the characteristic function

of disjunction correlates T to the distributions TT, TF, and FT, and F to FF. It is convenient to order the distributions of truth-values in the first columns of the truth-tables always in the same way; we adopt the most frequently used lexicographical order, with T preceding F (D₂).

D10-2. The *t*-th distribution of truth-values for the degree *n* (1 or 2) = D_t the truth-value or sequence of two truth-values here stated:

<i>t</i>	<i>t</i> -th DISTRIBUTION	
	for degree one	for degree two
1	T	TT
2	F	TF
3	—	FT
4	—	FF

If the order of the distributions is thus established by convention, we do not need the whole truth-table in order to describe a characteristic function. It is sufficient to state the truth-values in the order in which they occur in the second column. This sequence of truth-values for a connection is called its **characteristic** (comp. Wittgenstein [Tractatus] 4.442, [Syntax] § 57). Thus we see, for instance, from the two truth-tables below that the characteristic for negation is FT, that for disjunction is TTTF. We shall see soon (§ 12) that all extensional connections, and only these, have a truth-table and hence a characteristic function and a characteristic. The characteristics of the singular and binary extensional connections are listed in column (5) of the subsequent table. The connections are completely characterized by their characteristics, and hence may be defined with their help. Thus a connective will be called a sign of disjunction if it possesses the characteristic TTTF.

The table that follows contains, further, the following items, to be explained later. For every connection, as de-

finer by the characteristic in column (5), the name is given in column (2), the abbreviated name in column (1), a semantical name for the connective in column (4). The terms in columns (1), (2), and (4) correspond to those in the same columns in the table in § 3, but here they have no subscript 'C'. In this way the semantical terms of this table are distinguished from the syntactical terms of the previous table.

SEMANTICAL CONCEPTS OF PROPOSITIONAL CONNECTIONS AND CONNECTIVES
IN NTT

(1)	(2)	(3)	(4)	(5)
Connections		Connectives		
Abbreviation	Ordinary Name	Customary Symbol	Semantical Name	Characteristic
I. The four singular connections				
Conn ₁ ¹	tautology	~	b ₁	TT
Conn ₂ ¹	(identity)		b ₂	TF
Conn ₃ ¹	negation		b ₃ (neg)	FT
Conn ₄ ¹	contradiction		b ₄	FF
II. The sixteen binary connections				
Conn ₁ ²	tautology	V	c ₁	TTTT
Conn ₂ ²	disjunction		c ₂ (dis)	TTTF
Conn ₃ ²	(inverse implication)		c ₃	TTFT
Conn ₄ ²	(first component)	⊃	c ₄	TTFF
Conn ₅ ²	implication		c ₅ (imp)	TFTT
Conn ₆ ²	(second component)		c ₆	TFTF
Conn ₇ ²	equivalence	≡	c ₇ (equ)	TFFT
Conn ₈ ²	conjunction		c ₈ (con)	TFFF
Conn ₉ ²	exclusion			FTTT
Conn ₁₀ ²	(non-equivalence)		c ₁₀	FTTF
Conn ₁₁ ²	(negation of second)		c ₁₁	FTFT
Conn ₁₂ ²	(first alone)		c ₁₂	FTFF
Conn ₁₃ ²	(negation of first)		c ₁₃	FTTT
Conn ₁₄ ²	(second alone)		c ₁₄	FFTF
Conn ₁₅ ²	bi-negation		c ₁₅	FFFF
Conn ₁₆ ²	contradiction		c ₁₆	FFFF

Thus e.g. 'sign of disjunction_c in K ' is a syntactical concept based on the rules of deduction of PC₁ (D₃₋₂); on the other hand, 'sign of disjunction in S ' (D₁₁₋₂₃) is a semantical concept based on the rules of NTT, i.e. the truth-tables.

As examples of rules of NTT, stated in the customary form of diagrams (truth-tables), we give here those for negation and disjunction.

TRUTH-TABLE FOR NEGATION			TRUTH-TABLE FOR DISJUNCTION		
	\mathcal{S}_i	$\text{neg}(\mathcal{S}_i)$		\mathcal{S}_i \mathcal{S}_j	$\text{dis}(\mathcal{S}_i, \mathcal{S}_j)$
N1.	T	F	Dj1.	T T	T
N2.	F	T	Dj2.	T F	T
			Dj3.	F T	T
			Dj4.	F F	F

We may regard each line in a truth-table as a representation of a semantical rule. We designate the two semantical rules for negation, represented by the two lines of its normal truth-table, by 'N₁' and 'N₂', likewise the four rules for disjunction by 'Dj₁' to 'Dj₄', those for conjunction by 'C₁' to 'C₄', those for implication by 'I₁' to 'I₄', and those for equivalence by 'E₁' to 'E₄'. Formulated in words instead of diagrams, these rules are as follows.

+ **Rules of NTT** for some of the connections; \mathcal{S}_i and \mathcal{S}_j are any closed sentences.

N1. If \mathcal{S}_i is true, $\text{neg}(\mathcal{S}_i)$ is false.

N2. If \mathcal{S}_i is false, $\text{neg}(\mathcal{S}_i)$ is true.

Dj1. If \mathcal{S}_i and \mathcal{S}_j are true, $\text{dis}(\mathcal{S}_i, \mathcal{S}_j)$ is true.

Dj2. If \mathcal{S}_i is true and \mathcal{S}_j is false, $\text{dis}(\mathcal{S}_i, \mathcal{S}_j)$ is true.

Dj3. If \mathcal{S}_i is false and \mathcal{S}_j is true, $\text{dis}(\mathcal{S}_i, \mathcal{S}_j)$ is true.

Dj4. If \mathcal{S}_i and \mathcal{S}_j are false, $\text{dis}(\mathcal{S}_i, \mathcal{S}_j)$ is false.

In shorter formulation:

N1, 2. $\text{neg}(\mathcal{S}_i)$ is true if and only if \mathcal{S}_i is false.

Dj1 to 4. $\text{dis}(\mathcal{S}_i, \mathcal{S}_j)$ is true if and only if at least one of the two components is true.

Further:

C1 to 4. $\text{con}(\mathcal{S}_i, \mathcal{S}_j)$ is true if and only if both components are true.

I1 to 4. $\text{imp}(\mathcal{S}_i, \mathcal{S}_j)$ is true if and only if \mathcal{S}_i is false or \mathcal{S}_j is true or both.

E1 to 4. $\text{equ}(\mathcal{S}_i, \mathcal{S}_j)$ is true if and only if both components are true or both are false.

The rules of NTT for the other extensional connections can be formulated in an analogous way on the basis of their characteristics.

The following counter-examples show that it is necessary to restrict the application of the *truth-tables* to *closed sentences*. 1. Propositional logic. ' p ' and ' $\sim p$ ' are both false (this is often overlooked because of a confusion between propositional variables and propositional constants), in spite of N2. And ' $p \vee \sim p$ ' is true, in spite of Dj4. 2. Functional logic (customary interpretation, open sentences interpreted as universal, see § 28). If there is an individual which is not P, and another one which is P and not Q, then ' $P(x)$ ' and ' $\sim P(x)$ ' (which is L-equivalent to ' $(x)(\sim P(x))$ ') are both false, in spite of N2. Further, ' $Q(x)$ ' and ' $P(x) \supset Q(x)$ ' are false, in spite of I4. The necessity of the restriction is often not noticed. Sometimes the truth-tables are even explicitly formulated for open sentences, e.g. ' p ', ' q ', ' $p \vee q$ '. [It may be remarked that, in another sense, the truth-tables may be formulated for open sentences, e.g. propositional variables. While it is incorrect to formulate: "If ' p ' is false and ' q ' is false, then ' $p \equiv q$ ' is true", the following is correct: "If p is false and q is false, then $p \equiv q$ is true". In the latter sentence, ' p ', ' q ', and ' \equiv ' are regarded as belonging to the English language. The sentence refers to the absolute concept of truth for propositions ([I] § 17), not to the semantical concept of truth for sentences. Therefore it cannot serve as a rule for a language system.]

Suppose that a semantical system S contains a binary general connective (D2-1) a_k . Under what conditions shall we call a_k a sign of disjunction in S ? We shall not require

that S contain just the rules of NTT as formulated above. S may contain rules formulated in an entirely different way, provided only that they have the same effect upon the truth-value of a full sentence $\alpha_k(\mathcal{S}_i, \mathcal{S}_j)$ as the rules of NTT. Thus α_k will be called a sign of disjunction in S if, for any closed components \mathcal{S}_i and \mathcal{S}_j , $\alpha_k(\mathcal{S}_i, \mathcal{S}_j)$ is true if and only if at least one of the components is true, in other words, if it possesses the characteristic TTTF. Here, however, two cases must be distinguished. The condition mentioned may be fulfilled either (a) by accident, so to speak, or (b) necessarily. In case (a), the situation is such that we have to use factual knowledge, namely of the truth-values of the sentences involved, in order to find out whether the condition is fulfilled. In case (b), factual knowledge is not required; the rules of S suffice for showing that the condition is fulfilled. In case (b) we shall say that α_k has TTTF not only as a characteristic but also as an L-characteristic and that it is not only a sign of disjunction but also a sign of disjunction_L. (In case (a), we might call α_k a sign of disjunction_F.) Analogously for the other connections. On the basis of these considerations, we shall lay down definitions for the connections.

§ 11. The Connections in NTT

The concepts for the connections can easily be defined with the help of the characteristics. Thus, for instance, α_k is a sign of disjunction if it has the characteristic TTTF. We define, in addition, corresponding L-concepts. If the semantical rules suffice to show that α_k has the characteristic TTTF, then this is called its L-characteristic, and α_k is called a sign of disjunction_L. The definitions make use of the concept of L-range ([I] §§ 18 to 20).

Our task is now to formulate general definitions for 'characteristic', 'L-characteristic', '(sign for) Conn_r', and 'LConn_r'

in accordance with considerations in the preceding section. These definitions and, further, those for 'extensional' and 'L-extensional' (§ 12) can best be formulated if we make use of the concept of **L-range**. It has previously been shown ([I] § 19; see the example for S_4) how the semantical rules of a system S can be formulated as rules for L-range instead of rules for truth and how the L-concepts ('L-true', etc.) and the radical semantical concepts ('true', etc.) can be defined on the basis of 'L-range' ([I] § 20). Let us briefly summarize the main features of the previous explanations and definitions. By **L-states** with respect to S we mean either completely specified possible states of affairs of the objects dealt with in S ([I] § 18), or other entities corresponding to them, e.g. state-descriptions ([I] § 19). The L-range of a sentence \mathfrak{S}_m —designated by ' $\text{Lr}\mathfrak{S}_m$ ' or, in what follows, also by ' \mathfrak{R}_m '—is the class of those L-states which are admitted by \mathfrak{S}_m , i.e. those in which \mathfrak{S}_m would be true. If the rules of S are formulated as rules of L-range, then the concept 'L-range in S ' is defined by these rules, that is to say, for every \mathfrak{S}_m in S , $\text{Lr}\mathfrak{S}_m$ is determined by the rules. On this basis, the following concepts can be defined (as in [I] § 20). ' \mathfrak{V}_s ' designates the universal L-range, i.e. the class of all L-states, ' \mathfrak{A}_s ' the null L-range. The L-range of a sentential class \mathfrak{R}_i is the product of the L-ranges of the sentences of \mathfrak{R}_i . (For the signs of the theory of classes, e.g. ' ϵ ', ' \subset ', ' $+$ ', ' \times ', ' $-$ ', etc., see [I] § 6 at the end.)

D11-5. \mathfrak{T}_i is **L-true** (in S) $=_{\text{Df}}$ $\text{Lr}\mathfrak{T}_i = \mathfrak{V}_s$.

D11-6. \mathfrak{T}_i is **L-false** (in S) $=_{\text{Df}}$ $\text{Lr}\mathfrak{T}_i = \mathfrak{A}_s$.

D11-7. $\mathfrak{T}_i \xrightarrow{\text{L}} \mathfrak{T}_j$ (in S) $=_{\text{Df}}$ $\text{Lr}\mathfrak{T}_i \subset \text{Lr}\mathfrak{T}_j$.

D11-8. \mathfrak{T}_i is **L-equivalent** to \mathfrak{T}_j (in S) $=_{\text{Df}}$ $\text{Lr}\mathfrak{T}_i = \text{Lr}\mathfrak{T}_j$.

D11-9. \mathfrak{T}_i is **L-non-equivalent** to \mathfrak{T}_j (in S) $=_{\text{Df}}$ $\text{Lr}\mathfrak{T}_i = -\text{Lr}\mathfrak{T}_j$.

The definitions of radical concepts make use of 'rs' also, which designates the real L-state. While the concept of L-range, and hence also the other L-concepts just defined, are determined by the rules of *S*, this is not the case for rs and the radical concepts. In order to find out which L-state is the real one, factual knowledge is required.

D11-12. \mathfrak{R}_i is true (in *S*) =_{df} $rs \in Lr\mathfrak{R}_i$.

The other radical concepts are defined on the basis of 'true' in the customary way (the definitions in [I] § 20 are like those in [I] § 9).

T11-1. The L-range of \mathfrak{R}_i is the product of the L-ranges of the sentences of \mathfrak{R}_i . ([I] D20-1b.)

T11-6a [b]. (For 'a [b]', see remarks preceding D3-7.) Each of the following four conditions is sufficient and necessary for \mathfrak{R}_m and \mathfrak{R}_n to be [*L*]-equivalent in *S*.

1. $(R_m \times R_n) + (-R_m \times -R_n)$ contains rs [is V_s]. (From [I] D20-16, [I] T20-26 [D8].)
2. Both $R_m + (-R_n)$ and $-R_m + R_n$ (and hence also their product) contain rs [are V_s]. (From (1).)
3. $(R_m + R_n) \times (-R_m + (-R_n))$ does not contain rs [is Λ_s]. (From (1).)
4. Both $R_m \times -R_n$ and $-R_m \times R_n$ (and hence also their sum) do not contain rs [are Λ_s]. (From (3).)

T11-7a [b]. Each of the following five conditions is sufficient and necessary for \mathfrak{R}_m and \mathfrak{R}_n to be [*L*]-non-equivalent in *S*.

1. $(R_m \times -R_n) + (-R_m \times R_n)$ contains rs [is V_s]. (From [I] T9-21, [I] T20-26 [D9].)
2. Both $R_m + R_n$ and $-R_m + (-R_n)$ (and hence also their product) contain rs [are V_s]. (From (1).)

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3. $(R_m + (-R_n)) \times (-R_m + R_n)$ does not contain rs [is Λ_n]. (From (1).)
4. Both $R_m \times R_n$ and $-R_m \times -R_n$ (and hence also their sum) do not contain rs [are Λ_n]. (From (3).)
5. \mathfrak{T}_m and \mathfrak{T}_n are [L-]disjunct and [L-]exclusive. (From (2), [I] D20-17 [9], [I] D20-18 [10].)

Now we have to formulate the rules of NTT in terms of L-range. Let us take Dj2 as an example; we call it the second rule for disjunction (or Conn₂²) in NTT. It says: "If \mathfrak{S}_i is true and \mathfrak{S}_j is false, then $\text{dis}(\mathfrak{S}_i, \mathfrak{S}_j)$ is true." If, for a connective α_k , we find two sentences \mathfrak{S}_i and \mathfrak{S}_j such that \mathfrak{S}_i is true, \mathfrak{S}_j is false, and $\alpha_k(\mathfrak{S}_i, \mathfrak{S}_j)$ (which we will call \mathfrak{S}_k) true, then we say that α_k satisfies the rule Dj2 with respect to \mathfrak{S}_i and \mathfrak{S}_j . If, on the other hand, \mathfrak{S}_i is true, \mathfrak{S}_j is false, but \mathfrak{S}_k is false, then we say that α_k violates Dj2 with respect to \mathfrak{S}_i and \mathfrak{S}_j . If α_k satisfies Dj2 with respect to any closed components, then we say simply that α_k satisfies Dj2. The condition which in this case must be fulfilled for any closed \mathfrak{S}_i and \mathfrak{S}_j can also be stated in this way: "Either it is not the case that \mathfrak{S}_i is true and \mathfrak{S}_j is false, or \mathfrak{S}_k is true", or: " \mathfrak{S}_i is not true or \mathfrak{S}_j is not false or \mathfrak{S}_k is true". This is, in terms of L-range: " $\text{rs} \in -R_i$ or $\text{rs} \in R_j$ or $\text{rs} \in R_k$ ", or in other words: " $\text{rs} \in -R_i + R_j + R_k$ ". If, for some \mathfrak{S}_i and \mathfrak{S}_j , the class $-R_i + R_j + R_k$ is the universal L-range V_n , then we can know that it contains rs without knowing which L-state is rs. Thus, in this case, we know from the rules of S, without using factual knowledge, that α_k satisfies Dj2 with respect to \mathfrak{S}_i and \mathfrak{S}_j , and therefore we say that α_k L-satisfies Dj2 with respect to \mathfrak{S}_i and \mathfrak{S}_j . If this is the case for any closed \mathfrak{S}_i and \mathfrak{S}_j , we say that α_k L-satisfies Dj2. If α_k satisfies all four rules Dj1 to 4, then it has TTTF as a characteristic and is called a sign of Conn₂² or of disjunction in S. If α_k L-satisfies the four rules, then it has TTTF as an L-charac-

teristic and is called a sign of LConn_2^2 or of disjunction_L in S . On the basis of these considerations, we shall now lay down the general definitions for connections. Sometimes, when there is no danger of ambiguity, we write simply ' \mathfrak{S}_k ' for ' $a_k(\mathfrak{S}_i)$ ' or ' $a_k(\mathfrak{S}_i, \mathfrak{S}_j)$ ', i.e. the full sentence of a_k with the component or components under consideration, and hence ' R_k ' for the L-range of that full sentence.

D11-14. a_k satisfies the t -th rule ($t = 1$ to 4) in NTT for Conn_r^2 ($r = 1$ to 16) with respect to $\mathfrak{S}_i, \mathfrak{S}_j$ in $S =_{\text{Df}} a_k$ is a binary general connective in S ; \mathfrak{S}_i and \mathfrak{S}_j are closed and have the t -th distribution of truth-values (D10-2); $a_k(\mathfrak{S}_i, \mathfrak{S}_j)$ has the t -th truth-value in the characteristic of Conn_r^2 as given in column (6) of the table in § 10. Analogously for a singular connective.

D11-15. a_k violates the t -th rule ($t = 1$ to 4) in NTT for Conn_r^2 ($r = 1$ to 16) with respect to $\mathfrak{S}_i, \mathfrak{S}_j$ in $S =_{\text{Df}} a_k$ is a binary general connective in S ; \mathfrak{S}_i and \mathfrak{S}_j are closed and have the t -th distribution of truth-values; $a_k(\mathfrak{S}_i, \mathfrak{S}_j)$ does not have the t -th truth-value in the characteristic of Conn_r^2 . Analogously for a singular connective.

D11-16a [b]. (Auxiliary term for D17 and D21.) a_k has the [L-]characteristic value X (T or F) for the t -th distribution ($t = 1$ to 4) of *degree two* in $S =_{\text{Df}} a_k$ is a binary general connective in S ; and if \mathfrak{S}_i and \mathfrak{S}_j are any closed sentences in S and \mathfrak{S}_k is the full sentence $a_k(\mathfrak{S}_i, \mathfrak{S}_j)$, then the class specified below contains rs [is V_s]:

t	VALUE X	CLASS
1	T	$\neg R_i + (\neg R_j) + R_k$
1	F	$\neg R_i + (\neg R_j) + (\neg R_k)$
2	T	$\neg R_i + R_j + R_k$
2	F	$\neg R_i + R_j + (\neg R_k)$
3	T	$R_i + (\neg R_j) + R_k$
3	F	$R_i + (\neg R_j) + (\neg R_k)$
4	T	$R_i + R_j + R_k$
4	F	$R_i + R_j + (\neg R_k)$

Analogously for *degree one* ($t = 1$ or 2):

t	VALUE X	CLASS
1	T	$\neg R_i + R_k$
1	F	$\neg R_i + (\neg R_k)$
2	T	$R_i + R_k$
2	F	$R_i + (\neg R_k)$

T11-9a [b]. If a_k has the $[L-]$ characteristic value X (T or F) for the t -th distribution of degree two in S , and \mathcal{S}_i and \mathcal{S}_j are closed sentences which have the t -th distribution of $[L-]$ truth-values ($[L-]$ truth or $[L-]$ falsity), then $a_k(\mathcal{S}_i, \mathcal{S}_j)$ has the $[L-]$ truth-value X . Analogously for degree one.

Proof for $t = 1$, $X = T$, $\mathcal{S}_k = a_k(\mathcal{S}_i, \mathcal{S}_j)$. $\neg R_i + (\neg R_j) + R_k$ contains rs [is Λ_s] (D16a[b]). Since \mathcal{S}_i is $[L-]$ true, $\neg R_i$ does not contain rs [is V_s]; likewise $\neg R_j$. Therefore, R_k contains rs [is V_s]. Hence, \mathcal{S}_k is $[L-]$ true (D5). Analogously for the other values of t and X .

T11-10. a_k has the characteristic value X for the t -th distribution of degree two in S if and only if, for every closed \mathcal{S}_i and \mathcal{S}_j which have the t -th distribution of truth-values, $a_k(\mathcal{S}_i, \mathcal{S}_j)$ has the truth-value X . Analogously for degree one.

Proof. 1. T9a. 2. Proof for $t = 1$, $X = T$. Suppose that, for every closed \mathcal{S}_i and \mathcal{S}_j which have the first distribution and hence are true (D10-2), the full sentence \mathcal{S}_k has the value T. In other words, either \mathcal{S}_i and \mathcal{S}_j are not both true or \mathcal{S}_k is true; either \mathcal{S}_i is false or \mathcal{S}_j is false or \mathcal{S}_k is true; either $rs \in \neg R_i$ or $rs \in \neg R_j$ or $rs \in R_k$; $rs \in \neg R_i + (\neg R_j) + R_k$. Then a_k has the characteristic value T (D16a). Analogously for the other values of t and X .

D11-17a [b]. a_k $[L-]$ satisfies generally the t -th rule ($t = 1$ to 4) in NTT for the binary connection Conn_r^2 ($r = 1$ to 16) in $S =_{D1} a_k$ has an $[L-]$ characteristic value for the t -th distribution which is the same as the t -th value in the characteristic for Conn_r^2 , as given in column (6) of the table in § 10. Analogously for a singular connection ($t = 1$ and 2).

T11-11. α_k satisfies generally the t -th rule ($t = 1$ to 4) for Conn_r^2 ($r = 1$ to 16) in NTT if and only if, for every closed \mathcal{S}_i and \mathcal{S}_j in S which have the t -th distribution of truth-values, $\alpha_k(\mathcal{S}_i, \mathcal{S}_j)$ has the t -th truth-value in the characteristic of Conn_r^2 , and hence α_k satisfies the t -th rule with respect to $\mathcal{S}_i, \mathcal{S}_j$ in S . Analogously for degree one. (From D17a, T10, D14.)

D11-21a [b]. α_k has \mathcal{R}_k (a sequence of four truth-values) as its [L-]characteristic in $S =_{\text{Df}}$ one of the following conditions is fulfilled.

1. α_k is a singular general connective and has [L-]characteristic values for the first and second distribution of degree one, and \mathcal{R}_k is the sequence of these two truth-values in this order.
2. α_k is a binary general connective and has [L-]characteristic values for the first, second, third, and fourth distribution of degree two, and \mathcal{R}_k is the sequence of these truth-values in this order.

+**D11-23a [b].** α_k is a *sign for the connection* $[_L]\text{Conn}_r^n$ in $S =_{\text{Df}}$ α_k has as its [L-]characteristic the characteristic given in column (5) of the table in § 10 for Conn_r^n . Conn_1^1 is also called **tautology** (see column (2) of the table in § 10), Conn_3^1 **negation**, Conn_4^1 **contradiction**; further, Conn_1^2 **tautology**, Conn_2^2 **disjunction**, Conn_5^2 **implication**, Conn_7^2 **equivalence**, Conn_8^2 **conjunction**, Conn_9^2 **exclusion**, Conn_{15}^2 **bi-negation**, Conn_{16}^2 **contradiction**. Analogously, ${}_L\text{Conn}_3^1$ is also called **negation_L**, ${}_L\text{Conn}_2^2$ **disjunction_L**, etc.

It will be shown later (T12-28) that the signs for the connections $[_L]$ are [L-]extensional.

In some cases the term used for a connection is the same as that for a semantical relation. (In this point, our terminology at present follows the general use in spite of its disadvantages.) It is important to notice the difference between the two concepts, e.g. between the connection

of implication, for which there may be a sign (e.g. ' \supset ') in the object language, and the semantical relation of implication ($[I]$ D9-3), which is expressed in the metalanguage (e.g. by 'implies' or ' \rightarrow '), and likewise between the connection of equivalence (' \equiv ') and the semantical relation of equivalence ('equivalent to'). In the case of other connections, there is much less danger of confusion because, fortunately, different terms are used. Examples: the connection of disjunction (' \vee ') and the semantical relation of disjunctness ($[I]$ D9-5) ('disjunct'); the connection of negation (' \sim ') and the corresponding semantical property of falsity ('false'). — An analogous difference must be observed in the case of L-concepts; here we put the 'L' at different places. We must distinguish between the connection (sometimes we use here the term 'connection_L') of implication_L (' \supset '), introduced by its truth-table) and the (L-)semantical relation of L-implication ('L-implies', ' \rightarrow_L '); likewise between equivalence_L (' \equiv ') and L-equivalence ('L-equivalent to'), between disjunction_L (' \vee ') and L-disjunctness, between negation_L (' \sim ') and L-falsity ('L-false'). — Because of the danger of the confusion mentioned, it might be advisable to consider the use of other terms for the connections Conn₅² (implication) and Conn₇² (equivalence) (perhaps Quine's terms 'conditional' and 'biconditional'), and to reserve the terms 'implication' and 'equivalence' for the semantical relations.

We shall sometimes use ' b_q ' ($q = 1$ to 4) (see column (4) in the table in § 10) for a sign of Conn_q¹ (in most cases, for _LConn_q¹) and ' c_r ' ($r = 1$ to 16) for a sign of Conn_r² (in most cases, _LConn_r²). Instead of ' b_3 ', we usually write '**neg**' as the semantical name of a sign of negation, and '**neg_L**' as the name of a sign of negation_L; further, '**dis**', '**imp**', '**equ**', '**con**' for signs of disjunction, implication, equivalence, and conjunction respectively, '**dis_L**', '**imp_L**', '**equ_L**', '**con_L**' for signs of disjunction_L, implication_L, equivalence_L, and conjunction_L respectively. These names of connectives are mostly used for forming semantical descriptions of full sentences; '**neg_L(\mathfrak{S}_i)**', for instance, designates the full sentence of the sign of negation_L with \mathfrak{S}_i as component.

T11-12a [**b**]. a_k is a sign for $[_L]\text{Conn}_q^n$ ($n = 1$ or 2) in

S if and only if a_k [L-]satisfies generally all rules (two for $n = 1$, four for $n = 2$) for Conn_q^n in NTT. (From D23, D21, D17.)

The method here used for the definition of 'L-characteristic' (D21) and 'sign for LConn_q^n ' (D23) with the help of the concept of L-range, is analogous to that previously used in [Syntax] § 57 for the definition of corresponding concepts in syntax. In syntax, however, this method does not always lead to adequate concepts; this has been shown by Tarski (see "Addition 1935" at the end of § 57). Therefore the method is now transferred to L-semantics (compare [I] § 39, remarks on [Syntax] § 57).

T11-17a [b]. Let S contain a_k as a singular general connective and b_q ($q = 1$ to 4) as a sign for $[\text{L}] \text{Conn}_q^1$. a_k is also a sign for $[\text{L}] \text{Conn}_q^1$ if and only if, for any closed \mathfrak{S}_i , $a_k(\mathfrak{S}_i)$ is [L-]equivalent to $b_q(\mathfrak{S}_i)$. Analogously for a binary connective.

Proof for a[b]. b_q has the [L-]characteristic for Conn_q^1 (D23), say X_1X_2 . 1. If for any closed \mathfrak{S}_i , $\mathfrak{S}_k (= a_k(\mathfrak{S}_i))$ is [L-]equivalent to $\mathfrak{S}_q (= b_q(\mathfrak{S}_i))$, then both $R_k + (-R_q)$ and $-R_k + R_q$ contain rs [are V_s] (T6 (2)). If $X_1 = T$, then $-R_i + R_q$ contains rs [is V_s] (D16); hence likewise $(-R_i + R_q) + (R_k + (-R_q))$, which is $-R_i + R_k$. Therefore, a_k also has T as its [L-]characteristic value for the first distribution (D16). If $X_1 = F$, then $-R_i + (-R_q)$ contains rs [is V_s]; hence likewise $(-R_i + (-R_q)) + (-R_k + R_q)$, which is $-R_i + (-R_k)$. Therefore, a_k also has F as its value for the first distribution. Analogously for the second distribution ($X_2 = T$ or $X_2 = F$). Thus, a_k has the same two [L-]characteristic values as b_q , and hence the same [L-]characteristic (D21), and hence is also a sign for $[\text{L}] \text{Conn}_q^1$ (D23). — 2. If a_k is also a sign for $[\text{L}] \text{Conn}_q^1$, a_k has also the [L-]characteristic X_1X_2 . Take for example TF. Then, for any closed \mathfrak{S}_i , $-R_i + R_q$, $R_i + (-R_q)$, $-R_i + R_k$, and $R_i + (-R_k)$ all contain rs [are V_s] (D16); hence likewise $(-R_i + R_k) + (R_i + (-R_q))$, which is $R_k + (-R_q)$, and $(R_i + (-R_k)) + (-R_i + R_q)$, which is $-R_k + R_q$. Therefore, \mathfrak{S}_k and \mathfrak{S}_q are [L-]equivalent. Analogously for the other three characteristics (TT, FT, FF).

If, in constructing S , we lay down the rules of NTT for the four singular and the sixteen binary connectives, then we

see from these rules of S , without using factual knowledge, that those signs satisfy the rules of NTT and hence have the characteristics as given in column (6) of the table in § 10. Therefore, in this case, the signs L -satisfy the rules, have L -characteristics, and are signs for the connections $_L$. Since, however, the details of the formulation of the rules are inessential as long as they lead to the same results, we will also say that S contains NTT if the rules of S are formulated in any other way provided they give to the connectives the same properties as the rules of NTT would do, in other words, if they are such that they determine the same L -characteristics and hence make the connectives signs for the connections $_L$. This consideration leads to D26. For the sake of simplicity, we apply the definition only to systems whose sentences are all closed.

D11-26. S contains NTT = $_{Df}$ all sentences of S are closed; S contains a sign for each singulary or binary connection $_L$, i.e. four signs $_L b_q$ ($q = 1$ to 4) for $_L Conn^1$, and sixteen signs $_L c_r$ ($r = 1$ to 16) for $_L Conn^2$. These signs are called **connectives of NTT** in S .

If S contains NTT, we understand by the **ultimate components** of \mathcal{S}_i with respect to NTT those sentences out of which \mathcal{S}_i is constructed with the help of the connectives of NTT, which sentences themselves, however, are not full sentences of connectives of NTT. If \mathcal{S}_i is not a full sentence of a connective of NTT, \mathcal{S}_i is itself its only ultimate component. The ultimate components of \mathcal{R}_i are those of the sentences of \mathcal{R}_i . Thus for instance, the only ultimate component of $\text{dis}(\text{neg}(\mathcal{S}_1), \mathcal{S}_1)$ is \mathcal{S}_1 . If in this example \mathcal{S}_1 is replaced by any other sentence, the resulting sentence is always L -true (see below T13-25b(2)); therefore, $\text{dis}(\text{neg}(\mathcal{S}_1), \mathcal{S}_1)$, and likewise $\text{dis}(\text{neg}(\mathcal{S}_i), \mathcal{S}_i)$ for any \mathcal{S}_i , is L -true by NTT. Thus this concept applies to those sentences which can be shown to be true and hence L -true by merely apply-

ing the normal truth-tables for the connectives occurring. Analogously, we shall define 'L-false by NTT', 'L-implies by NTT', and 'L-equivalent by NTT' in such a way that these concepts apply to those cases where the rules of NTT suffice to show that the corresponding radical concepts and hence the L-concepts hold. For technical reasons, we first define 'L-implies by NTT' by reference to the replacements, and then the other concepts on the basis of this concept. T25 shows that these definitions are in agreement with the previous definitions for the L-concepts (D5 to 8).

D11-29. \mathfrak{X}_i L-implies \mathfrak{X}_j by NTT in $S =_{\text{Df}}$ S contains NTT; $\mathfrak{X}'_i \xrightarrow{L} \mathfrak{X}'_j$ in S for any \mathfrak{X}'_i and \mathfrak{X}'_j which are constructed out of \mathfrak{X}_i and \mathfrak{X}_j respectively in the following way: any ultimate components of \mathfrak{X}_i and of \mathfrak{X}_j are replaced by any sentences in S ; if a component occurs at several places in \mathfrak{X}_i and \mathfrak{X}_j , then it must be replaced, if at all, by the same sentence at all places where it occurs in \mathfrak{X}_i and \mathfrak{X}_j .

D11-30. \mathfrak{X}_i is L-true by NTT in $S =_{\text{Df}}$ $\Lambda \xrightarrow{L} \mathfrak{X}_i$ by NTT.

D11-31. \mathfrak{X}_i is L-false by NTT in $S =_{\text{Df}}$ $\mathfrak{X}_i \xrightarrow{L} V$ by NTT.

D11-32. \mathfrak{X}_i is L-equivalent to \mathfrak{X}_j by NTT in $S =_{\text{Df}}$ $\mathfrak{X}_i \xrightarrow{L} \mathfrak{X}_j$ by NTT, and $\mathfrak{X}_j \xrightarrow{L} \mathfrak{X}_i$ by NTT.

T11-24. If S contains NTT, then the following holds:

- a. An infinite number of sentences in S are L-true by NTT.
- b. An infinite number of sentences in S are L-false by NTT.
- c. V is L-false by NTT.

Proof. a. Given any sentence \mathfrak{S}_m which is L-true by NTT (e.g. for any \mathfrak{S}_i , $\text{dis}_L(\text{neg}_L(\mathfrak{S}_i), \mathfrak{S}_i)$), its double negation (i.e. $\text{neg}_L(\text{neg}_L(\mathfrak{S}_m))$) is also L-true by NTT. — b. Given any sentence \mathfrak{S}_n which is L-false in NTT (e.g. $\text{con}_L(\mathfrak{S}_i, \text{neg}_L(\mathfrak{S}_i))$), its double negation is also L-false in NTT. — c. From (b), [I] T14-11.

T11-25.

- a. If $\mathfrak{X}_i \xrightarrow{L} \mathfrak{X}_j$ by NTT (in S), then $\mathfrak{X}_i \xrightarrow{L} \mathfrak{X}_j$.
- b. If \mathfrak{X}_i is L-true by NTT, then \mathfrak{X}_i is L-true.
- c. If \mathfrak{X}_i is L-false by NTT, then \mathfrak{X}_i is L-false.
- d. If \mathfrak{X}_i is L-equivalent to \mathfrak{X}_j by NTT, then \mathfrak{X}_i is L-equivalent to \mathfrak{X}_j .

Proof. a. From D29, with replacement of the components by themselves. — b. From (a), [I] T14-51a. — c. From (a), [I] P14-7. — d. From (a).

§ 12. Extensionality

A connection or connective is usually called extensional (or a truth-function) if the truth-value of its full sentences depends merely upon the truth-values of the components. We also define the corresponding L-concept (with the help of the concept of L-range) in such a way that a connective is L-extensional if the semantical rules suffice to show that it is extensional. A connective is extensional if and only if it has a characteristic; L-extensional if (but not only if) it has an L-characteristic. Thus the connections listed in the table in § 10 are extensional and under certain conditions L-extensional.

The connections which have truth-tables are often called *truth-functions*, because the truth-value of a full sentence depends merely upon the truth-values of the components. Following Russell, we call connections of this kind and their connectives **extensional** and a full sentence of such a connection extensional with respect to the components. For a singular connective α_k , the condition of extensionality can be formulated in the following way: if \mathfrak{S}_i and \mathfrak{S}'_i are any closed sentences which have the same truth-value and hence are equivalent, then $\alpha_k(\mathfrak{S}_i)$ and $\alpha_k(\mathfrak{S}'_i)$, which we will call \mathfrak{S}_k and \mathfrak{S}'_k respectively, also have the same truth-value and hence are equivalent. As in the case of many other well-known concepts, we introduce here a distinction between an L- and an F-concept dependent upon the distinction be-

tween the case where the general condition for the concept in question, here the condition for extensionality just mentioned, is fulfilled by the contingency of the facts, and the case where it is fulfilled necessarily, that is to say, in such a way that we can find out that it is fulfilled on the basis of the semantical rules of the system in question without using factual knowledge. In order to make this distinction, we transform the condition of extensionality in the following way: "Either \mathcal{S}_i and \mathcal{S}'_i are not equivalent or \mathcal{S}_k and \mathcal{S}'_k are equivalent"; further, in terms of L-range (where ' R'_i ' is short for ' $Lr\mathcal{S}'_i$ '): "Either $rs \in ((R_i \times -R'_i) + (-R_i \times R'_i))$ or $rs \in ((R_k \times R'_k) + (-R_k \times -R'_k))$ " (T₁₁-7a(1), T₁₁-6a(1)); " $rs \in ((R_i \times -R'_i) + (-R_i \times R'_i) + (R_k \times R'_k) + (-R_k \times -R'_k))$ ". In general, factual knowledge is required in order to find out that the last-mentioned class contains rs . Only in the case that this class is V_s can we know that it contains rs without knowing which L-state is rs . In this case, we call α_k **L-extensional**. If the condition for extensionality is not fulfilled, we call the connective **non-extensional**. [The term 'intensional' is often used in this case. Since, however, this term is used in traditional logic in another sense, it might be advisable not to use it here.] If the class mentioned is null, then we know without the use of factual knowledge that it does not contain rs and that hence the connective is non-extensional; we call it in this case **L-non-extensional**. These considerations lead to the following definitions (D₁ and 2); the definitions for binary connections (D₃ and 4) are analogous.

Conditions for extensionality, non-extensionality, and L-non-extensionality which do not make use of the concept of L-range are given in T₁ and T_{2a,b} below. It seems that a condition, and hence a definition, for L-extensionality cannot be given in this simple way (except in special cases where S contains certain connectives; see below, T₁₃-31b). This is the chief reason for applying the concept of L-range in D_{1b} and D_{3b}.

D12-1a [b]. A *singular* connection (and a general connective for it) is **[L-]extensional** in $S =_{Df}$ if \mathcal{S}_i and \mathcal{S}'_i are closed and \mathcal{S}_k is the full sentence with \mathcal{S}_i , and \mathcal{S}'_i that with \mathcal{S}'_i , then $(R_i \times -R'_i) + (-R_i \times R'_i) + (R_k \times R'_k) + (-R_k \times -R'_k)$ contains rs [is V_s].

D12-2a [b]. A *singular* connection (and a general connective for it) is **[L-]non-extensional** (intensional) in $S =_{Df}$ there are \mathcal{S}_i , \mathcal{S}'_i , \mathcal{S}_k , and \mathcal{S}'_k such that \mathcal{S}_k is the full sentence with \mathcal{S}_i and \mathcal{S}'_i that with \mathcal{S}'_i , and each of the following four classes (and hence also their sum) does not contain rs [is Λ_s]: $R_i \times -R'_i$, $-R_i \times R'_i$, $R_k \times R'_k$, $-R_k \times -R'_k$.

D12-3a [b]. A *binary* connection (and a general connective for it) is **[L-]extensional** in $S =_{Df}$ if \mathcal{S}_i , \mathcal{S}'_i , \mathcal{S}_j , and \mathcal{S}'_j are closed and \mathcal{S}_k is the full sentence with \mathcal{S}_i and \mathcal{S}_j , and \mathcal{S}'_i and \mathcal{S}'_j , then $(R_i \times -R'_i) + (-R_i \times R'_i) + (R_j \times -R'_j) + (-R_j \times R'_j) + (R_k \times R'_k) + (-R_k \times -R'_k)$ contains rs [is V_s]. The class mentioned can also be stated in the following form: $((R_i + R'_i) \times (-R_i + -R'_i)) + ((R_j + R'_j) \times (-R_j + -R'_j)) + ((R_k + -R'_k) \times (-R_k + R'_k))$.

D12-4a [b]. A *binary* connection (and a general connective for it) is **[L-]non-extensional** (intensional) in $S =_{Df}$ there are \mathcal{S}_i , \mathcal{S}'_i , \mathcal{S}_j , \mathcal{S}'_j , \mathcal{S}_k , \mathcal{S}'_k such that \mathcal{S}_k is the full sentence with \mathcal{S}_i and \mathcal{S}_j , and \mathcal{S}'_i and \mathcal{S}'_j , and each of the following six classes (and hence also their sum) does not contain rs [is Λ_s]: $R_i \times -R'_i$, $-R_i \times R'_i$, $R_j \times -R'_j$, $-R_j \times R'_j$, $R_k \times R'_k$, $-R_k \times -R'_k$.

The following theorem, **T₁**, states the condition for extensionality in the customary form. There is no analogue to this theorem concerning L-extensionality (see, above, the remark preceding **D₁**).

+T12-1. A singular general connective α_k is *extensional* in S if and only if the following holds. If \mathcal{S}_i and \mathcal{S}'_i are any

closed equivalent sentences, then $\alpha_k(\mathcal{S}_i)$ is equivalent to $\alpha_k(\mathcal{S}'_i)$. (From D1a, T11-7a(1), T11-6a(1).) Analogously for a binary connective.

T12-2a [b]. Let α_k be a singular general connective in S . Each of the following conditions, applying to every closed \mathcal{S}_i and \mathcal{S}'_i with the full sentences \mathcal{S}_k and \mathcal{S}'_k , is a sufficient and necessary condition for α_k to be $[L]$ -extensional.

1. $-Q_i + Q_k$ contains rs [is V_α , or in other words, $Q_i \subset Q_k$]. Here, $Q_i = (R_i \times R'_i) + (-R_i \times -R'_i) = (R_i + (-R'_i)) \times (-R_i + R'_i)$; hence, $-Q_i = (R_i \times -R'_i) + (-R_i \times R'_i) = (R_i + R'_i) \times (-R_i + (-R'_i))$; $Q_k = (R_k \times R'_k) + (-R_k \times -R'_k) = (R_k + (-R'_k)) \times (-R_k + R'_k)$.
2. Each of the following four classes, and hence also their product, contains rs [is V_α]: $R_i + R'_i + R_k + (-R'_k)$, $R_i + R'_i + (-R_k) + R'_k$, $-R_i + (-R'_i) + R_k + (-R'_k)$, $-R_i + (-R'_i) + (-R_k) + R'_k$.

T12-3a [b]. A singular general connective α_k is $[L]$ -non-extensional in S if and only if there are closed sentences \mathcal{S}_i , \mathcal{S}'_i such that \mathcal{S}_i and \mathcal{S}'_i are $[L]$ -equivalent, and $\alpha_k(\mathcal{S}_i)$ and $\alpha_k(\mathcal{S}'_i)$ are $[L]$ -non-equivalent. (From D2, T11-6(4), T11-7(4).) Analogously for a binary connective.

T12-4. Let α_k be a singular general, L -extensional connective. If \mathcal{S}_i and \mathcal{S}'_i are closed and L -equivalent, $\alpha_k(\mathcal{S}_i)$ and $\alpha_k(\mathcal{S}'_i)$ are L -equivalent. Analogously for a binary connective.

Proof. Let \mathcal{S}_k and \mathcal{S}'_k be the full sentences. If \mathcal{S}_i and \mathcal{S}'_i are L -equivalent, $R_i = R'_i$ (D11-8). Since α_k is L -extensional, the class mentioned in D1b is V_α , hence also $(R_k \times R'_k) + (-R_k \times -R'_k)$. Therefore, $R_k = R'_k$; hence \mathcal{S}_k and \mathcal{S}'_k are L -equivalent.

T12-6. If a connective α_k in S satisfies a certain rule for

a connection in NTT with respect to some components and violates the same rule with respect to others, then α_k is non-extensional.

Proof. Let α_k be a singularly connective (the proof for a binary one is analogous). Let α_k satisfy the t -th rule for Conn_t^1 with respect to \mathcal{S}_i and violate the same rule with respect to \mathcal{S}_i' . Then (D11-14 and 15) \mathcal{S}_i and \mathcal{S}_i' have the t -th distribution of truth values and hence are equivalent, while the full sentences have not the same truth-value and hence are non-equivalent. Therefore, α_k is non-extensional (T3a).

T12-12a [b]. Let S contain at least one [L-]true sentence and at least one [L-]false sentence and α_k be a singularly general connective. Then the following holds:

1. For any t (1 or 2), α_k has at most one t -th [L-]characteristic value (D11-16).
2. α_k has at most one [L-]characteristic.

Analogously for a binary connective.

Proof for a [b] (1). Let \mathcal{S}_i be [L-]true and \mathcal{S}_i' [L-]false in S . Let \mathcal{S}_k and \mathcal{S}_k' be the full sentences. If for $t = 1$, α_k had both T and F as an [L-]characteristic value, then \mathcal{S}_k would be both [L-]true and [L-]false, which is impossible. (More exactly, on the basis of our definitions in terms of 'L-range': both $-R_i + R_k$ and $-R_i + (-R_k)$ would contain rs [be V_*] (D11-16a[b]); since \mathcal{S}_i is [L-]true, R_i contains rs [is V_*]; hence $-R_i$ does not contain rs [is Λ_*]; hence both R_k and $-R_k$ would contain rs [be V_*], which is impossible. Analogously for $t = 2$, with \mathcal{S}_i' . — a[b](2) from a[b](1).

T12-13. Let S contain at least one true and at least one false sentence and α_k be a singularly general connective. If α_k is extensional, then it has one and only one characteristic. The same holds for a binary connective.

Proof. Let \mathcal{S}_1 be true and \mathcal{S}_2 false in S , and α_k be extensional. Let \mathcal{S}_{k1} be $\alpha_k(\mathcal{S}_1)$, and \mathcal{S}_{k2} $\alpha_k(\mathcal{S}_2)$. We distinguish two cases: 1. \mathcal{S}_{k1} is true; 2. it is false. 1. For any closed \mathcal{S}_i , either \mathcal{S}_i is false or \mathcal{S}_i is true and hence equivalent to \mathcal{S}_1 , and hence $\alpha_k(\mathcal{S}_i)$, which we call \mathcal{S}_{ki} , is equivalent to \mathcal{S}_{k1} (T1) and hence also true. Therefore, rs $\epsilon -R_i + R_{k1}$. Since this holds for any \mathcal{S}_i , α_k has the characteristic value T for the first distribution of degree one (D11-16a). 2. It can be shown

analogously that, if \mathcal{S}_{k1} is false, α_k has the characteristic value F for the first distribution. It can be shown in an analogous way with \mathcal{S}_2 that α_k has T or F as its characteristic value for the second distribution. Therefore, α_k has a characteristic (D11-21a) but not more than one (T12a (2)).

It is to be noted that no strict analogue to T13 holds for L-concepts. Let α_k be L-extensional. Then it is extensional and hence has a characteristic (T13). It does not, however, necessarily have an L-characteristic. Since α_k is L-extensional, the semantical rules without factual knowledge suffice to show that α_k has one of the characteristics; but they do not necessarily suffice to find out which one. And only if they do is that characteristic an L-characteristic for α_k . If they do not, then α_k , though L-extensional, has no L-characteristic.

Example of an L-extensional connective without L-characteristic. Let α_k be a singular general connective in S . Let W be the condition that Mt. Washington is less than 4000 ft. high, and R_w be the class of those L-states in which W holds. Let the following rule of L-range be laid down for the full sentence \mathcal{S}_k of α_k with any component \mathcal{S}_i : $R_k = (R_i \times R_w) + (-R_i \times -R_w)$. Thus $rs \in R_k$ if and only if either $rs \in R_i$ and $rs \in R_w$, or not $rs \in R_i$ and not $rs \in R_w$. Thus \mathcal{S}_k is true if either \mathcal{S}_i is true and W holds, or \mathcal{S}_i is false and W does not hold. Hence, if \mathcal{S}_i means (designates the proposition) A , $\alpha_k(\mathcal{S}_i)$ means: A if and only if W . Let \mathcal{S}_i and \mathcal{S}'_i be equivalent and \mathcal{S}'_k be $\alpha_k(\mathcal{S}'_i)$. Then either both \mathcal{S}_i and \mathcal{S}'_i have the same truth-value as W , in which case \mathcal{S}_k and \mathcal{S}'_k are both true; or both have a truth-value different from that of W , in which case \mathcal{S}_k and \mathcal{S}'_k are both false. Thus, in any case, the full sentences are equivalent. Therefore, α_k is extensional (Tr). Since this has been found without factual knowledge, α_k is L-extensional. [This latter reasoning is rather vague. The same result is shown in the following more exact way on the basis of D1b. The rule given for R_k holds generally. Therefore, $R'_k = (R'_i \times R_w) + (-R'_i \times -R_w)$. By substituting these values for R_k and R'_k in the expression of a class in D1, after a simple transformation we find that class to be $(R_i \times -R'_i) + (-R_i \times R'_i) + (R_i \times R'_i) + (-R_i \times -R'_i)$, which is V_* . Hence, α_k is L-extensional (D1b).] Without using factual

knowledge, we know that α_k has a characteristic, but we do not know which one. By empirical investigation, measuring the height of Mt. Washington, we find that W does not hold. It follows that every full sentence of α_k with a true component is false, while that with a false component is true. Hence, α_k has the characteristic FT and is a sign of negation. But FT is not an L -characteristic of α_k , and α_k is not a sign of negation $_L$. (If we wish to use F -terms, we might say that α_k has an F -characteristic and is a sign of negation $_F$.)

T12-16a [b]. Let every sentence in S be $[L]$ -true. Then for any singular general connective α_k in S , the following holds:

1. α_k has T and not F as its $[L]$ -characteristic value for the first distribution.
2. α_k has (vacuously) both T and F as $[L]$ -characteristic values for the second distribution (which does not occur).
3. α_k $[L]$ -satisfies generally the first rules for Conn_1^1 and Conn_2^1 (see the table in § 10).
4. The second rule for any singular connection is not applicable and hence is generally $[L]$ -satisfied by α_k .
5. α_k has both TT and TF as $[L]$ -characteristics.
6. α_k is a sign for both $[_L]\text{Conn}_1^1$ and $[_L]\text{Conn}_2^1$.
7. S contains no sign for Conn_3^1 or Conn_4^1 (and hence none for $_L\text{Conn}_3^1$ or $_L\text{Conn}_4^1$).
8. α_k is $[L]$ -extensional.

Proof for a $[b]$. 1 and 2. For any \mathcal{S}_i , with the full sentence \mathcal{S}_k , both \mathcal{S}_i and \mathcal{S}_k are $[L]$ -true. Hence both R_i and R_k contain rs [are V_*], while both $\neg R_i$ and $\neg R_k$ do not contain rs [are Λ_*]. Hence, $\neg R_i + R_k$, $R_i + R_k$, and $R_i + (\neg R_k)$ contain rs [are V_*], but $\neg R_i + (\neg R_k)$ does not. Hence (D11-16a [b]), α_k has T , but not F , as its $[L]$ -characteristic value for the first distribution and both T and F for the second. 3. From (1), D11-17a [b]. 4. From (2). 5. From (1), (2), D11-21a [b]. 6. From (5), D11-23a [b]. 7. From (1), D11-23a [b]. 8a. All sentences of S are equivalent to one another, hence also all full sentences of α_k . Hence, α_k is extensional (Tr).

8b. All sentences of S are L-true and hence have the L-range V_s ; hence also any full sentences \mathfrak{S}_k and \mathfrak{S}'_k . Therefore, $R_k \times R'_k = V_s$, and hence likewise the class mentioned in D1. Therefore, a_k is L-extensional (D1b).

The proofs for the following theorems, T17, 20, and 21, are analogous to that for T16.

T12-17a [b]. Let every sentence in S be [L]-true. Then for any binary general connective a_k in S , the following holds:

1. a_k has T and not F as its [L]-characteristic value for the first distribution.
2. a_k has (vacuously) both T and F as [L]-characteristic values for the second, third, and fourth distributions (which do not occur).
3. a_k [L]-satisfies generally the first rules for the eight connections Conn_1^2 to Conn_8^2 (i.e. those whose characteristic begins with T).
4. The second, third, and fourth rules for any binary connection are not applicable and hence are generally [L]-satisfied by a_k .
5. a_k has simultaneously those eight [L]-characteristics which begin with T.
6. a_k is a sign simultaneously for the eight connections ${}_{[L]}\text{Conn}_1^2$ to ${}_{[L]}\text{Conn}_8^2$.
7. S contains no sign for any of the connections Conn_9^2 to Conn_{16}^2 (and hence none for the corresponding connections_L).
8. a_k is [L]-extensional.

T12-20a [b]. Let every sentence in S be [L]-false. Then for any singulary general connective a_k in S the following holds:

1. a_k has F and not T as [L]-characteristic value for the second distribution.
2. a_k has (vacuously) both T and F as [L]-charac-

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teristic values for the first distribution (which does not occur).

3. a_k [L-]satisfies generally the second rules for Conn_3^1 and Conn_4^1 .
4. The first rule for any singular connection is not applicable and hence is generally [L-]satisfied by a_k .
5. a_k has both TF and FF as [L-]characteristics.
6. a_k is a sign for both $[_L]\text{Conn}_2^1$ and $[_L]\text{Conn}_4^1$.
7. S contains no sign for Conn_1^1 or Conn_3^1 (and hence none for the corresponding connections_L).
8. a_k is [L-]extensional.

T12-21a [b]. Let every sentence in S be [L-]false. Then for any binary general connective a_k in S the following holds:

1. a_k has F and not T as its [L-]characteristic value for the fourth distribution.
2. a_k has (vacuously) both T and F as [L-]characteristic values for the first, second, and third distributions (which do not occur).
3. a_k [L-]satisfies generally the fourth rules for the eight connections Conn_r^2 with even r (i.e. those whose characteristic ends with F).
4. The first, second, and third rules for any binary connection are not applicable and hence are generally [L-]satisfied by a_k .
5. a_k has simultaneously those eight [L-]characteristics which end with F.
6. a_k is a sign simultaneously for the eight connections $[_L]\text{Conn}_r^2$ with even r .
7. S contains no sign for any of the connections Conn_r^2 with odd r (and hence none for the corresponding connections_L).
8. a_k is [L-]extensional.

+**T12-25a** [b]. If α_k is a singular or binary general connective in S and has an [L-]characteristic, then α_k is [L-]extensional.

Proof for a [b]. Let α_k be a singular general connective (the proof for a binary connective is analogous) which has an [L-]characteristic. Let \mathfrak{S}_i and \mathfrak{S}'_i be any closed sentences with the full sentences \mathfrak{S}_k and \mathfrak{S}'_k respectively. Let the four classes mentioned in T2(2) be k_1, k_2, k_3 , and k_4 . If the first truth-value in the [L-]characteristic of α_k is T, both $\neg R_i + R_k$ and $\neg R'_i + R'_k$ contain rs [are V_s] (D11-16a [b]), and hence both k_3 and k_4 contain rs [are V_s]. Likewise, if the first value is F, then $\neg R_i + (\neg R_k)$, $\neg R'_i + (\neg R'_k)$, k_4 , and k_3 contain rs [are V_s]. If the second value is T, then $R_i + R_k$, $R'_i + R'_k$, k_1 , and k_2 contain rs [are V_s]. If the second value is F, then $R_i + (\neg R_k)$, $R'_i + (\neg R'_k)$, k_2 , and k_1 contain rs [are V_s]. Thus, in the case of each of the possible [L-]characteristics TT, TF, FT, and FF, each of the classes k_1 to k_4 contains rs [is V_s]. Hence α_k is [L-]extensional (T2a [b](2)).

+**T12-26**. If α_k is a singular or binary extensional connective, then α_k has a characteristic. (From T13, T16a(5), T17a(5).)

This is the converse of T25a. An analogue to T26 for L-concepts, which would be the converse of T25b, does not hold (see the remark on T13 and the counter-example).

T12-28a [b]. If α_k is a sign for $[_L]\text{Conn}^1_q$ ($q = 1$ to 4) or for $[_L]\text{Conn}^2_r$ ($r = 1$ to 16), then α_k is [L-]extensional. (From D11-23, T25.)

§ 13. Theorems Concerning Particular Connections

Some theorems concerning negation, disjunction, conjunction, implication, equivalence, and the corresponding connections_L (negation_L, etc.) are stated. Some of these theorems state sufficient and necessary conditions for a sign to be a connective for one of these connections (T₅, T₁₃, T₁₄); the L-ranges of full sentences (T₁₅); relations between the connections_L and certain radical [L-]concepts, e.g. [L-]true (T₂₀, T₂₅ to 28); sufficient and necessary conditions for [L-]extensionality (T₃₁) and [L-]non-extensionality (T₃₂).

T13-3a [b]. α_k [L-]satisfies generally the rule N₁ for negation in NTT in S if and only if α_k is a singular general connective in S , and for any closed \mathfrak{S}_i with the full sentence \mathfrak{S}_k the following condition (stated in three forms) is fulfilled:

1. $-R_i + (-R_k)$ contains rs [is V_s]. (From D_{II-17a} [b], D_{II-16a} [b].)
2. $R_i \times R_k$ does not contain rs [is Λ_s]. (From (1).)
3. \mathfrak{S}_i and \mathfrak{S}_k are [L-]exclusive. (From (1) [(2)], [I] D₂₀₋₁₈[10].)

T13-4a [b]. α_k [L-]satisfies generally the rule N₂ in NTT in S if and only if α_k is a singular general connective in S , and for any closed \mathfrak{S}_i with the full sentence \mathfrak{S}_k the following condition (stated in three forms) is fulfilled:

1. $R_i + R_k$ contains rs [is V_s]. (From D_{II-17a} [b], D_{II-16a} [b].)
2. $-R_i \times -R_k$ does not contain rs [is Λ_s]. (From (1).)
3. \mathfrak{S}_i and \mathfrak{S}_k are [L-]disjunct. (From (1), [I] D₂₀₋₁₇[9].)

+T13-5a [b]. α_k is a *sign of negation*_[L] in S if and only if α_k is a singular general connective in S , and for any

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closed \mathfrak{S}_i with the full sentence \mathfrak{S}_k the following condition (stated in three forms) is fulfilled:

1. Both $R_i + R_k$ and $-R_i + (-R_k)$ (and hence also their product) contain rs [are V_s , and hence $R_k = -R_i$]. (From T_{11-12} , $T_3(1)$, $T_4(1)$.)
2. \mathfrak{S}_i and \mathfrak{S}_k are [L-]disjunct and [L-]exclusive. (From $T_3(3)$, $T_4(3)$.)
3. \mathfrak{S}_i and \mathfrak{S}_k are [L-]non-equivalent [and hence, $R_k = -R_i$]. (From (2), $T_{11-7}(5)$ [and D_{11-9}].)

T13-10a [b] (lemma). A binary general connective a_k in S [L-]satisfies generally one of the rules in NTT mentioned below (Dj_1 to 4 for disjunction, C_1 to 4 for conjunction, I_1 to 4 for implication, E_1 to 4 for equivalence) if and only if, for any closed sentences \mathfrak{S}_i and \mathfrak{S}_j with the full sentence \mathfrak{S}_k , the class specified below for that rule contains rs [is V_s]. (From D_{11-23} , D_{11-21} , D_{11-17} , D_{11-16} .)

	RULE	CLASS
1.	Dj_1	$-R_i + (-R_j) + R_k$
2.	Dj_2	$-R_i + R_j + R_k$
3.	Dj_3	$R_i + (-R_j) + R_k$
4.	Dj_4	$R_i + R_j + (-R_k)$
5.	C_1	$-R_i + (-R_j) + R_k$
6.	C_2	$-R_i + R_j + (-R_k)$
7.	C_3	$R_i + (-R_j) + (-R_k)$
8.	C_4	$R_i + R_j + (-R_k)$
9.	I_1	$-R_i + (-R_j) + R_k$
10.	I_2	$-R_i + R_j + (-R_k)$
11.	I_3	$R_i + (-R_j) + R_k$
12.	I_4	$R_i + R_j + R_k$
13.	E_1	$-R_i + (-R_j) + R_k$
14.	E_2	$-R_i + R_j + (-R_k)$
15.	E_3	$R_i + (-R_j) + (-R_k)$
16.	E_4	$R_i + R_j + R_k$

+**T13-13a [b]**. A binary general connective a_k is a sign for $_{[L]}Conn^r$ ($r = 2, 5, 7, 8$) in S if and only if, for every closed \mathfrak{S}_i and \mathfrak{S}_j with the full sentence \mathfrak{S}_k , the two classes specified below (and hence also their product) contain rs

[are V_a , or, in other words, if and only if the condition A stated below is fulfilled]. (From T11-12, T10.)

r	CONNECTION	CLASSES	A
2	disjunction	$-(R_i + R_j) + R_k$ and $R_i + R_j + (-R_k)$	$R_k = R_i + R_j$
5	implication	$(R_i \times -R_j) + R_k$ and $-R_i + R_j + (-R_k)$	$R_k = -R_i + R_j$
7	equivalence	$(R_i \times -R_j) + (-R_i \times R_j) + R_k$ and $(R_i \times R_j) + (-R_i \times -R_j) + (-R_k)$	$R_k = (R_i \times R_j) + (-R_i \times -R_j)$
8	conjunction	$-(R_i \times R_j) + R_k$ and $(R_i \times R_j) + (-R_k)$	$R_k = R_i \times R_j$

+T13-14a [b]. A binary general connective a_k is a sign of conjunction_[L] in S if and only if, for every closed \mathfrak{S}_i and \mathfrak{S}_j , $a_k(\mathfrak{S}_i, \mathfrak{S}_j)$ is [L-]equivalent to $\{S_i, S_j\}$. (From T13(8), [I] D20-1b, T11-6(2).)

The following theorem states the L-ranges of full sentences of some connections_L in terms of the L-ranges of the components.

+T13-15. For any closed \mathfrak{S}_i and \mathfrak{S}_j in S , the following holds if S contains the connectives referred to:

1. $Lr(\text{neg}_L(\mathfrak{S}_i)) = -R_i$. (From T5b(1).)
2. $Lr(\text{dis}_L(\mathfrak{S}_i, \mathfrak{S}_j)) = R_i + R_j$. (From T13b(2).)
3. $Lr(\text{imp}_L(\mathfrak{S}_i, \mathfrak{S}_j)) = -R_i + R_j$. (From T13b(5).)
4. $Lr(\text{equ}_L(\mathfrak{S}_i, \mathfrak{S}_j)) = (R_i \times R_j) + (-R_i \times -R_j)$. (From T13b(7).)
5. $Lr(\text{con}_L(\mathfrak{S}_i, \mathfrak{S}_j)) = R_i \times R_j$. (From T13b(8).)

+T13-20a [b]. For any closed \mathfrak{S}_i and \mathfrak{S}_j in S , the following holds if S contains the connectives referred to. [If S contains NTT, then the relations stated hold also for the L-concepts by NTT (D11-29 to 32).]

1. $\text{neg}_{[L]}(\mathfrak{S}_i)$ is [L-]true if and only if \mathfrak{S}_i is [L-]false.
2. $\text{neg}_{[L]}(\mathfrak{S}_i)$ is [L-]false if and only if \mathfrak{S}_i is [L-]true.

3. $\text{dis}_{[L]}(\mathcal{S}_i, \mathcal{S}_j)$ is $[L]$ -true if and only if \mathcal{S}_i and \mathcal{S}_j are $[L]$ -disjunct.
4. $\text{dis}_{[L]}(\mathcal{S}_i, \mathcal{S}_j)$ is $[L]$ -false if and only if both \mathcal{S}_i and \mathcal{S}_j are $[L]$ -false.
5. $\text{con}_{[L]}(\mathcal{S}_i, \mathcal{S}_j)$ is $[L]$ -true if and only if both \mathcal{S}_i and \mathcal{S}_j (and hence $\{\mathcal{S}_i, \mathcal{S}_j\}$) are $[L]$ -true.
6. $\text{con}_{[L]}(\mathcal{S}_i, \mathcal{S}_j)$ is $[L]$ -false if and only if \mathcal{S}_i and \mathcal{S}_j are $[L]$ -exclusive.
7. $\text{imp}_{[L]}(\mathcal{S}_i, \mathcal{S}_j)$ is $[L]$ -true if and only if $\mathcal{S}_i \xrightarrow{[L]} \mathcal{S}_j$.
8. $\text{imp}_{[L]}(\mathcal{S}_i, \mathcal{S}_j)$ is $[L]$ -false if and only if \mathcal{S}_i is $[L]$ -true and \mathcal{S}_j $[L]$ -false.
9. $\text{equ}_{[L]}(\mathcal{S}_i, \mathcal{S}_j)$ is $[L]$ -true if and only if \mathcal{S}_i and \mathcal{S}_j are $[L]$ -equivalent.
10. $\text{equ}_{[L]}(\mathcal{S}_i, \mathcal{S}_j)$ is $[L]$ -false if and only if \mathcal{S}_i and \mathcal{S}_j are $[L]$ -non-equivalent (and hence $[L]$ -disjunct and $[L]$ -exclusive).

Proof. (a) can easily be shown with the help of the characteristics of the connections involved, or, in other words, with the rules of NTT. — Proof for (b). Let the full sentence in each case be \mathcal{S}_k . 1. \mathcal{S}_k is L -true if and only if R_k is V_* (D11-5), hence if and only if R_i is Λ_* (T5b(1)), hence if and only if R_i is L -false (D11-6). 2. Likewise from D11-6, T5b(1), D11-5. 3. \mathcal{S}_k is L -true if and only if $R_k = V_*$, hence if and only if $R_i + R_j = V_*$ (T13b), hence if and only if \mathcal{S}_i and \mathcal{S}_j are L -disjunct ($[I]$ D20-9). 4. \mathcal{S}_k is L -false if and only if $R_i + R_j = \Lambda_*$ (D11-6, T13b(2)), hence if and only if both R_i and R_j are Λ_* , hence if and only if both \mathcal{S}_i and \mathcal{S}_j are L -false. (5) from T14b. (6) from T14b, $[I]$ D20-10. 7. \mathcal{S}_k is L -true if and only if $-R_i + R_j = V_*$ (T15(3)), hence if and only if $R_i \subset R_j$, hence if and only if $\mathcal{S}_i \xrightarrow{L} \mathcal{S}_j$ (D11-7). 8. \mathcal{S}_k is L -false if and only if $-R_i + R_j = \Lambda_*$ (T15(3)), hence if and only if $R_i = V_*$ and $R_j = \Lambda_*$, hence if and only if \mathcal{S}_i is L -true and \mathcal{S}_j L -false. 9. \mathcal{S}_k is L -true if and only if $(R_i \times R_j) + (-R_i \times -R_j) = V_*$ (T15(4)), hence if and only if \mathcal{S}_i and \mathcal{S}_j are L -equivalent (T11-6b(1)). 10. \mathcal{S}_k is L -false if and only if $(R_i \times R_j) + (-R_i \times -R_j) = \Lambda_*$ (T15(4)), hence if and only if \mathcal{S}_i and \mathcal{S}_j are L -non-equivalent (T11-7b(4)), and hence L -disjunct and L -exclusive (T11-7b(5)). — Since the relations hold for any components \mathcal{S}_i and \mathcal{S}_j , they hold also for the L -concepts by NTT.

If S contains signs for the connections_[L] involved, then theorem T20 gives sufficient and necessary conditions for certain radical, L-, and F-concepts, namely for (L-, F-) falsity (1), disjunctness (3), implication (7), and equivalence (9). This is the method of the so-called characteristic sentences, which has been previously explained ([I] § 22, especially T22-1 to 4) but can be exactly formulated only now that definitions for the connections have been given.

The parts (a) of the following theorems T25 to T28 are quite elementary and well-known. They can easily be proved with the help of truth-tables (i.e. on the basis of the semantical rules of NTT). Therefore we refer in the proofs to the parts (b) only.

T13-25a [b]. If S contains the connectives referred to, each of the following sentences is $[L]$ -true in S for any closed \mathcal{S}_m [and, moreover, L-true by NTT, if S contains NTT]:

1. $\text{dis}_{[L]}(\mathcal{S}_m, \text{neg}_{[L]}(\mathcal{S}_m))$.
2. $\text{dis}_{[L]}(\text{neg}_{[L]}(\mathcal{S}_m), \mathcal{S}_m)$.

Proof for b. For each of the sentences stated, the L-range can easily be found to be V_+ , with the help of T15. Therefore, the sentence is L-true (D11-5). For example, the L-range for (1) is found to be $R_m + -R_m$, which is V_+ . If S contains NTT, then each of the sentences is, moreover, L-true by NTT (D11-30) because it is L-true for any \mathcal{S}_m .

T13-26a [b]. If S contains the connectives referred to, then in each of the following cases $\mathcal{I}_i [L]$ -implies \mathcal{I}_j for any closed \mathcal{S}_m and \mathcal{S}_n [and, moreover, \mathcal{I}_i L-implies \mathcal{I}_j by NTT if S contains NTT].

	\mathcal{I}_i	\mathcal{I}_j
1.	\mathcal{S}_m	$\text{dis}_{[L]}(\mathcal{S}_m, \mathcal{S}_n)$
2.	\mathcal{S}_n	$\text{dis}_{[L]}(\mathcal{S}_m, \mathcal{S}_n)$
3.	$\text{con}_{[L]}(\mathcal{S}_m, \mathcal{S}_n)$	\mathcal{S}_m
4.	$\text{con}_{[L]}(\mathcal{S}_m, \mathcal{S}_n)$	\mathcal{S}_n

Proof for b. In each case, by determining the L-ranges with the help of T15 and T11-1, it is easily found that $R_i \subset R_j$; therefore,

$\mathfrak{I}_i \xrightarrow{L} \mathfrak{I}_j$ (D11-7). For instance, in (1), $R_i = R_m$, $R_j = R_m + R_n$ (T15(2)). For L-implication by NTT, see D11-29.

T13-27a [b]. If S contains the connectives referred to, then in each of the following cases \mathfrak{I}_i and \mathfrak{I}_j are $[L]$ -equivalent for any closed \mathfrak{S}_m and \mathfrak{S}_n [and, moreover, L-equivalent by NTT if S contains NTT].

	\mathfrak{I}_i	\mathfrak{I}_j
1.	$\text{neg}_{[L]}(\text{neg}_{[L]}(\mathfrak{S}_m))$	\mathfrak{S}_m
2.	$\text{imp}_{[L]}(\mathfrak{S}_m, \mathfrak{S}_n)$	$\text{dis}_{[L]}(\text{neg}_{[L]}(\mathfrak{S}_m), \mathfrak{S}_n)$
3.	$\text{con}_{[L]}(\mathfrak{S}_m, \mathfrak{S}_n)$	$\{\mathfrak{S}_m, \mathfrak{S}_n\}$

Proof for b. In each case with the help of T15 and T11-1, it is found that $R_i = R_j$; therefore, \mathfrak{I}_i and \mathfrak{I}_j are L-equivalent (D11-8). For instance, in (1), $R_i = -(-R_m) = R_m$, $R_j = R_m$. For L-equivalence by NTT, see D11-32.

T13-28a [b]. If S contains the connectives referred to, the sentence $\text{con}_{[L]}(\mathfrak{S}_m, \text{neg}_{[L]}(\mathfrak{S}_m))$ is $[L]$ -false for any closed \mathfrak{S}_m [and, moreover, L-false by NTT if S contains NTT].

Proof for b. The L-range of the sentence is $R_m \times -R_m$, hence Λ , (T15). Therefore the sentence is L-false (D11-6). For L-falsity by NTT, see D11-31.

T13-31a [b]. Let S contain a singular general connective a_k and signs of equivalence $_{[L]}$ and implication $_{[L]}$. Then each of the following conditions is a necessary and sufficient condition for a_k to be $[L]$ -extensional:

1. For any closed \mathfrak{S}_i and \mathfrak{S}'_i with the full sentences \mathfrak{S}_k and \mathfrak{S}'_k , $\text{equ}_{[L]}(\mathfrak{S}_i, \mathfrak{S}'_i) \xrightarrow{[L]} \text{equ}_{[L]}(\mathfrak{S}_k, \mathfrak{S}'_k)$.
2. For any \mathfrak{S}_i , \mathfrak{S}'_i , \mathfrak{S}_k , and \mathfrak{S}'_k as in (1), $\text{imp}_{[L]}(\text{equ}_{[L]}(\mathfrak{S}_i, \mathfrak{S}'_i), \text{equ}_{[L]}(\mathfrak{S}_k, \mathfrak{S}'_k))$ is $[L]$ -true.

(1) holds analogously for a binary connective, with $\{\text{equ}_{[L]}(\mathfrak{S}_i, \mathfrak{S}'_i), \text{equ}_{[L]}(\mathfrak{S}_j, \mathfrak{S}'_j)\} \xrightarrow{[L]} \text{equ}_{[L]}(\mathfrak{S}_k, \mathfrak{S}'_k)$.

Proof. a(1). a_k is extensional if and only if \mathfrak{S}_i and \mathfrak{S}'_i are not equivalent or \mathfrak{S}_k and \mathfrak{S}'_k are equivalent (T1), hence if and only

if $\text{equ}(\mathcal{S}_i, \mathcal{S}'_i)$ is false (T20a(10)) or $\text{equ}(\mathcal{S}_k, \mathcal{S}'_k)$ is true (T20a(9)), hence if and only if $\text{equ}(\mathcal{S}_i, \mathcal{S}'_i) \rightarrow \text{equ}(\mathcal{S}_k, \mathcal{S}'_k)$ ([I] (D9-3)). — $b(1)$. a_k is L-extensional if and only if $((R_i \times R'_i) + (-R_i \times -R'_i)) \subset ((R_k \times R'_k) + (-R_k \times -R'_k))$ (T12-2b(1)), hence if and only if $\text{Lr}(\text{equ}_L(\mathcal{S}_i, \mathcal{S}'_i)) \subset \text{Lr}(\text{equ}_L(\mathcal{S}_k, \mathcal{S}'_k))$ (T14b(4)), hence if and only if $\text{equ}(\mathcal{S}_i, \mathcal{S}'_i) \xrightarrow{L} \text{equ}(\mathcal{S}_k, \mathcal{S}'_k)$ (D11-7). — $a[b](2)$ from $a[b](1)$, T20a[b](7).

T13-32a [b]. Let S contain a singular general connective a_k and a sign of equivalence_[L]. a_k is [L]-non-extensional if and only if there are closed \mathcal{S}_i and \mathcal{S}'_i , with full sentences of a_k \mathcal{S}_k and \mathcal{S}'_k , such that $\text{equ}_{[L]}(\mathcal{S}_i, \mathcal{S}'_i)$ is [L]-true and $\text{equ}_{[L]}(\mathcal{S}_k, \mathcal{S}'_k)$ is [L]-false. (From T12-3, T20(9) and (10).) Analogously for a binary connective.

T13-35a [b]. Let S contain a sign of negation_[L]. Then, for any closed \mathcal{S}_i and \mathcal{S}'_i with full sentences of negation_[L] \mathcal{S}_k and \mathcal{S}'_k , the following holds:

1. $\mathcal{S}'_k \xrightarrow{[L]} \mathcal{S}_k$ if and only if $\mathcal{S}_i \xrightarrow{[L]} \mathcal{S}'_i$.
2. \mathcal{S}_k and \mathcal{S}'_k are [L]-equivalent if and only if \mathcal{S}_i and \mathcal{S}'_i are [L]-equivalent.

Proof. $a(1)$. From rules N1 and 2 in NTT, and [I] D9-3. — $b(1)$. From T15(1), D11-7. — (2) from (1).

T13-38a [b]. Let S contain a sign of disjunction_[L]. For any closed \mathcal{S}_i , \mathcal{S}'_i , \mathcal{S}_j in S , if $\mathcal{S}_i \xrightarrow{[L]} \mathcal{S}'_i$ then $\text{dis}_{[L]}(\mathcal{S}_i, \mathcal{S}_j) \xrightarrow{[L]} \text{dis}_{[L]}(\mathcal{S}'_i, \mathcal{S}_j)$. (a. From rules Dj1 to 4 in NTT. — b. From T15(2), D11-7.)

T13-39. Let S contain a sign of disjunction_L. Let $\mathcal{S}'_i, \mathcal{S}_j$ and the sentences of \mathcal{R}_i be closed. Let \mathcal{R}_k be the class constructed out of \mathcal{R}_i by replacing each sentence \mathcal{S}_m in \mathcal{R}_i by $\text{dis}_L(\mathcal{S}_m, \mathcal{S}_j)$. If $\mathcal{R}_i \xrightarrow{L} \mathcal{S}'_i$, then $\mathcal{R}_k \xrightarrow{L} \text{dis}_L(\mathcal{S}'_i, \mathcal{S}_j)$.

Proof. R_k is the product of the classes $R_m + R_j$, one for each sentence \mathcal{S}_m in \mathcal{R}_i (T15(2)). Hence, $R_k = R_i + R_j$. If $\mathcal{R}_i \xrightarrow{L} \mathcal{S}'_i$, $R_i \subset R'_i$ (D11-7), hence $R_i + R_j \subset R'_i + R_j$, hence $\mathcal{R}_k \xrightarrow{L} \text{dis}_L(\mathcal{S}'_i, \mathcal{S}_j)$.

C. INTERPRETATIONS OF PC

The possibilities of true interpretations for PC are examined. The system NTT is an L-true interpretation for PC. It is called the normal interpretation for PC (§ 14). The analysis leads to the result that there are two kinds of true non-normal interpretations for PC (§§ 15–17). Therefore, PC is not a full formalization of propositional logic (§ 18).

§ 14. NTT as an L-true Interpretation for PC

The well-known fact that the two customary methods for dealing with the propositional connectives — PC and NTT — lead to the same results is here formulated and proved in our terminology. It is shown that, under certain conditions, a system *S* containing NTT is an L-true interpretation for a calculus *K* containing PC (T₄).

In this chapter, C, we shall discuss the possible true interpretations of PC, or, more precisely, of the propositional connectives of PC. We shall leave aside here the problem of the interpretation of the propositional variables, although with respect to C-extensional calculi ([I] D₃₁₋₁₈), which are most frequently used, it is rather simple. [For the semantics of variables in general, see [I] § 11. It is planned to discuss the problem of interpretations for propositional variables in a later volume, in connection with the discussion of extensional and non-extensional systems.] Therefore, the discussion in this chapter will refer only to calculi containing PC without propositional variables. [As we have seen in § 10, the truth-tables apply only to closed sentences and therefore not to forms of PC with propositional variables.] For the sake of simplicity, we refer in this chapter only to the forms PC₁ and PC₁^D. On the basis of the definitions in § 4, the results hold likewise for any other form of PC.

The following theorem, T₁, says in effect that the definitions of other connectives on the basis of neg_c and dis_c in PC_1^D , as described in the table in § 3, are in agreement with NTT.

T14-1 (lemma). Let K contain PC_1 . Let S contain the sentences of K and contain NTT (D11-26) in such a way that the signs neg_c (' \sim ') and dis_c (' \vee ') in K are simultaneously the signs of negation_L and disjunction_L in NTT in S . Let \mathcal{S}_q^1 be the sentence given for $c\text{Conn}_q^1$ ($q = 1$ to 4) in column (5) of the table of connections in § 3, and likewise \mathcal{S}_r^2 that for $c\text{Conn}_r^2$ ($r = 1$ to 16), the components \mathcal{S}_i and \mathcal{S}_j being any closed sentences in K . Then \mathcal{S}_q^1 and $b_q(\mathcal{S}_i)$ are L-equivalent by NTT in S , and likewise \mathcal{S}_r^2 and $c_r(\mathcal{S}_i, \mathcal{S}_j)$.

This theorem is well-known. It can easily be verified by showing with the help of truth-tables that in each case the two sentences have the same L-characteristic. On the basis of our definitions, it can be shown by determining the L-ranges of the two sentences; it turns out that they are identical, and hence the sentences L-equivalent (D11-8). Thus e.g. for $r = 5$, $\text{Lr}(\text{dis}(\mathcal{S}_i, \mathcal{S}_j)) = -R_i + R_j$ (T13-15(1) and (2)); and likewise $\text{Lr}(\text{imp}(\mathcal{S}_i, \mathcal{S}_j)) = -R_i + R_j$ (T13-15(3)). Since the two sentences are L-equivalent whatever the components \mathcal{S}_i and \mathcal{S}_j may be, they are L-equivalent by NTT (D11-32).

The following theorem, T₂, says in effect that the rules of deduction in PC_1^D are in agreement with NTT. [It is to be noted that the conditions involve that all sentences in S be closed (C, D11-26), and hence also all sentences in K (B).]

T14-2 (lemma). Let K and S fulfill the following conditions:

- A. K contains PC_1 or PC_1^D .
- B. All sentences in K belong to S .
- C. S contains NTT in such a way that the sign for a connection_c of PC in K is simultaneously the sign for the corresponding connection_L of NTT in S (i.e. $c b_q$ in $K = b_q$ in S ($q = 1$ to 4), and $c c_r$ in $K = c_r$ in S ($r = 1$ to 16)).

Then the following holds:

- a. If \mathcal{S}_i is a primitive sentence in K in virtue of (the primitive sentence schemata of) PC_1 , then \mathcal{S}_i is L-true by NTT in S .
- b. If $\mathcal{R}_k \xrightarrow{dC} \mathcal{S}_j$ in virtue of the rule of inference of PC_1 , then $\mathcal{R}_k \xrightarrow{L} \mathcal{S}_j$ by NTT in S .
- c. If $\mathcal{S}_i \xrightarrow{dC} \mathcal{S}_j$ in virtue of one of the definition rules of PC_1^D , then $\mathcal{S}_i \xrightarrow{L} \mathcal{S}_j$ by NTT in S .

The proofs are well-known. They can easily be given by an analysis of each of the rules of deduction in PC_1^D . They are usually given on the basis of the truth-tables NTT. On the basis of our definitions, they are given by determining the L-ranges with the help of T13-15(1) and (2).

- a. For every primitive sentence, the L-range is V_* . — b. If $\mathcal{R}_k \xrightarrow{dC} \mathcal{S}_j$ according to D2-2b(5), then $R_k = R_i \times (-R_i + R_j)$ (T11-1) $= R_i \times R_j$; hence $R_k \subset R_j$; hence $\mathcal{R}_k \xrightarrow{L} \mathcal{S}_j$ (D11-7). — c. From T1. — The L-concepts hold by NTT (D11-30 and 29) because they hold for any components.

+T14-3. Let K and S fulfill the three conditions (A) to (C) in T2. Then the following holds:

- a. If $\mathcal{I}_i \xrightarrow{C} \mathcal{I}_j$ in K by PC (i.e. either PC_1 or PC_1^D), $\mathcal{I}_i \xrightarrow{L} \mathcal{I}_j$ by NTT in S . (From T2.)
- b. If \mathcal{I}_i is C-true in K by PC, \mathcal{I}_i is L-true by NTT in S . (From (a).)

+T14-4. Let K and S fulfill the following conditions:

- A, B, C, as in T2.
- D. K does not contain other rules of deduction than those of PC_1 or PC_1^D .

Then S is an *L-true interpretation* for K .

Proof. From T3a and [I] D34-1, because K does not contain rules of refutation (D).

+T14-5. Let K and S fulfill the following conditions:

- A, B, C, as in T2.
- D. K contains all sentences of S .

Let \mathfrak{R}_i be non-empty and finite, and \mathfrak{T}_i and \mathfrak{T}_j be finite. Then the following holds:

- a. If \mathfrak{S}_j is L-true by NTT in S , then \mathfrak{S}_j is provable and hence C-true by PC (D4-2) in K .
- b. If $\mathfrak{R}_i \xrightarrow{L} \mathfrak{S}_j$ by NTT in S , then \mathfrak{S}_j is derivable from \mathfrak{R}_i and hence a C-implicate of \mathfrak{R}_i by PC in K .
- c. If $\mathfrak{T}_i \xrightarrow{L} \mathfrak{T}_j$ by NTT in S , then $\mathfrak{T}_i \xrightarrow{C} \mathfrak{T}_j$ by PC in K .

Proof. a. As is well-known, a proof for \mathfrak{S}_j in K under the conditions stated can be constructed with the help of the conjunctive normal form (see Hilbert [Logik] Kap. I, § 3). — b. Let \mathfrak{S}_i be a sentence constructed so as to be C-equivalent to \mathfrak{R}_i by PC in K (e.g. a conjunction_C of the sentences of \mathfrak{R}_i). Then \mathfrak{S}_i is L-equivalent to \mathfrak{R}_i by NTT in S (T3a). Hence, if the condition in (b) is fulfilled, $\mathfrak{S}_i \xrightarrow{L} \mathfrak{S}_j$ by NTT. Therefore, $\text{imp}_L(\mathfrak{S}_i, \mathfrak{S}_j)$ is L-true by NTT in S (T13-20b(7)). Hence likewise $\text{dis}_L(\text{neg}_L(\mathfrak{S}_i), \mathfrak{S}_j)$ is L-true by NTT in S (T13-27b(2)) and C-true in K (a). It is the same sentence as $\text{dis}_C(\text{neg}_C(\mathfrak{S}_i), \mathfrak{S}_j)$ (C). Therefore, $\mathfrak{S}_i \xrightarrow{C} \mathfrak{S}_j$ in K (T7-1), and $\mathfrak{R}_i \xrightarrow{C} \mathfrak{S}_j$ in K . — c. If \mathfrak{T}_i is a non-empty, finite \mathfrak{R}_i , and \mathfrak{T}_j is \mathfrak{S}_j , the assertion is the same as (b). If \mathfrak{T}_i is Λ , and \mathfrak{T}_j is \mathfrak{S}_j , and $\mathfrak{T}_i \xrightarrow{L} \mathfrak{T}_j$ by NTT, then \mathfrak{T}_j is L-true by NTT in S (D11-30) and hence C-true in K (a); therefore $\mathfrak{T}_i \xrightarrow{C} \mathfrak{T}_j$ in K . If \mathfrak{T}_j is \mathfrak{R}_j , and $\mathfrak{T}_i \xrightarrow{L} \mathfrak{T}_j$ by NTT, then for every sentence \mathfrak{S}_k of \mathfrak{R}_j , $\mathfrak{T}_i \xrightarrow{L} \mathfrak{S}_k$ by NTT, and hence $\mathfrak{T}_i \xrightarrow{C} \mathfrak{S}_k$ in K . Therefore $\mathfrak{T}_i \xrightarrow{C} \mathfrak{T}_j$ in K ([I] T29-40).

T5c shows that, under the conditions stated, K is an *L-exhaustive calculus* for S ([I] D36-3) as far as C-implication among finite \mathfrak{T} is concerned (i.e. leaving aside infinite classes and L-falsity).

In the customary terminology, PC is said to be a *complete* calculus. This is sometimes meant in the sense that every sentence which is L-true by NTT ("tautology") is C-true ("provable") (T5a), sometimes in the sense that, in the form of PC with propositional variables as the only ultimate components (see § 2 at the end and § 4 at the end), every sentence is either C-true ("provable") or C-comprehen-

sive (usually called "refutable"). For proofs of completeness in the one or the other sense, see Quine, *Journ. Symb. Log.*, vol. 3, 1938, pp. 37ff, and his references to other authors: Post (1921), Hilbert and Ackermann (1928), Łukasiewicz (1931), Hilbert and Bernays (1934), Kalmar (1935), Hermes and Scholz (1937). We shall see later that, in a stricter sense of completeness, PC is not complete.

§ 15. Non-Normal Interpretations of Signs of Negation_C and Disjunction_C

The concepts of normal and L-normal interpretations for the connectives in a calculus are defined with the help of NTT (D1). It is shown that, under certain conditions, if a calculus contains two signs for the same connection_C and the first has a normal or L-normal interpretation, then the second has, too (T1 and 2). (This result might mislead us into the erroneous assumption that non-normal interpretations are impossible.) A non-normal interpretation of a connective would involve the violation of a truth-table. Therefore, the consequences of supposed violations of the single rules in NTT for disjunction (Dj1 to 4, § 10) and negation (N1 and 2) are examined. Some of the results: Dj1, 2, and 3 are generally satisfied (T4); if N1 is once violated, then it is always violated and all sentences are true (T5); if N2 is once violated, then the sign of negation_C is non-extensional (T7); if Dj4 is once violated, then the signs of disjunction_C and negation_C are non-extensional (T8).

We have already defined the syntactical concepts of signs for connections_C in a calculus, e.g. 'sign of disjunction_C in *K*', and the semantical concepts of signs for connections in a semantical system, e.g. 'sign of disjunction (or disjunction_L) in *S*'. Now we shall define a related concept which — like the concept of interpretation — refers both to a calculus and to a semantical system and hence belongs neither to syntax nor to semantics but to the combined field which we have called the theory of systems (compare [I] § 5 at the end and § 37). If, for instance, a_k is a sign of disjunction_C in *K* and simultaneously a sign of disjunction (or of disjunction_L)

in a true (or L-true) interpretation S for K , then we shall say that a_k has a normal (or L-normal, respectively) interpretation in S . D1 formulates this for connections in general.

+D15-1a [b]. The connective a_k in K has an [L-]normal interpretation in $S =_{D1} S$ is an [L-]true interpretation for K ; a_k is a sign for ${}_c\text{Conn}_q^1$ or ${}_c\text{Conn}_q^2$ in K (D4-3) and simultaneously a sign for the corresponding connection ${}_{[L]}\text{Conn}_q^1$ or ${}_{[L]}\text{Conn}_q^2$ in S (D11-23a [b]).

We shall now show that under certain conditions, if one sign for a certain connection in K has an [L-]normal interpretation in S , then the same holds for any other sign for the same connection in K (T1 and 2).

+T15-1a [b]. Let K fulfill the following conditions:

A, as in T8-9 (two signs of negation $_c$, neg_{c1} and neg_{c2}).

B and C, as in T6-10.

Then the following holds: If neg_{c1} has an [L-]normal interpretation in S , then also neg_{c2} .

Proof for a [b]. Let the conditions be fulfilled and \mathcal{S}_i be a closed sentence in K . Then $\text{neg}_{c1}(\mathcal{S}_i)$ and $\text{neg}_{c2}(\mathcal{S}_i)$ are C-equivalent in K (T8-9a). Since S is an [L-]true interpretation for K (D1), the two sentences are [L-]equivalent in S ([I], T33[34]-8g). Since neg_{c1} has an [L-]normal interpretation, it is a sign of negation $_{[L]}$ in S (D1). Therefore, neg_{c2} is also a sign of negation $_{[L]}$ in S (T11-17) and hence has an [L-]normal interpretation.

+T15-2a [b]. Let K fulfill the following conditions:

A. K contains two sub-calculi K_m and K_n , both containing PC_1^D .

B and C, as in T6-10.

D. a_m and a_n are connectives for the same connection $_c$ of PC_1^D in K_m and K_n , respectively.

Then the following holds: If a_m in K has an [L-]normal interpretation in S , then a_n likewise. (From T9-4a, T11-17, in analogy to T1.)

We shall now study the question of the possibility of non-normal interpretations for the connectives of PC. If we try to answer this question without closer investigation, we might be tempted to guess a negative answer. It will be shown that for conjunction_c a non-normal interpretation is indeed impossible. And we might perhaps believe that if a non-normal interpretation for another connection were possible, then in a calculus containing two connectives for this connection one could be interpreted normally and the other non-normally. Our previous result that this latter case cannot occur (T₁ and 2) might thus lead us to the assumption that non-normal interpretations are impossible. These considerations, however, turn out to be erroneous; we shall find non-normal interpretations.

Let K contain PC_1 or PC_1^D and α_k be a sign for the connection ${}_c\text{Conn}_i^2$ in K . Let S be a true interpretation of K such that the following is the case (provided this is possible; that will be discussed later): α_k is not a sign for Conn_i^2 in S and hence has a *non-normal interpretation* in S (D₁). Then at least one rule for Conn_i^2 , represented by a line in the truth-table for this connection, will be violated by α_k in S in at least one instance, i.e. with respect to at least one pair of closed sentences as components. This violation of a normal truth-table by α_k is not necessarily such that α_k has another truth-table in S . Let us suppose that a certain rule for Conn_i^2 in NTT states the value F for the value distribution TF of the components. Then it may happen that for some instance with the values TF the full sentence of α_k is indeed false, while for another instance with the same values TF it is true. If this happens, α_k has no truth-table in S , neither the normal nor another one; the truth-value of a full sentence of α_k is not a function of the truth-values of the components; α_k is *non-extensional* (D₁₂₋₂, T₁₂₋₆).

In order to find possible non-normal interpretations for

signs of negation_C and disjunction_C, we shall now study the possibilities of a violation for each of the rules for disjunction (Dj1 to 4) and negation (N1 and 2) in NTT, and analyze the consequences of these violations. It will first be shown that Dj1, 2, and 3 cannot be violated (T4). Then Dj4, N1, and N2 will be analyzed.

As has been remarked previously, we must distinguish between the concepts 'sign of negation_C in *K*' and 'sign of negation (or negation_L) in *S*', the first being syntactical, the second and third semantical. This distinction is of especial importance in the cases now to be studied, where rules of NTT are violated. If a sign of negation_C in *K* violates in *S* one of the rules N1 and N2, then it is not a sign of negation in *S*.

For some of the theorems in this and the following sections, we state two procedures for the proof, marked by 'I' and 'II'. Procedure I is rather simple; it is based on the formulation of the rules of NTT (e.g. Dj1 to 4) as given in § 10. Procedure II is more exact and more technical; it is based on the definitions in § 11 in terms of L-range. I applies only to radical concepts; if a theorem refers both to radical and to L-concepts (usually by 'a [*b*]'), then II applies to both. The results concerning non-normal interpretations will be chiefly in radical terms. Therefore a reader who is chiefly interested in those results and not in the general theory of true and L-true interpretations of PC, and who wants to travel an easy road to these results without technicalities, may skip part II in the proofs.

T15-4a [*b*]. Let *K* contain PC₁. Let *S* be any [*L*]true interpretation for *K*. Then disc_C in *K* [*L*]satisfies generally the rules Dj1, 2, and 3 of NTT.

Proof for a [*b*]. Let \mathcal{S}_i and \mathcal{S}_j be any closed sentences in *K*, and \mathcal{S}_k be disc_C($\mathcal{S}_i, \mathcal{S}_j$). Then \mathcal{S}_k is a C-implicate in *K* both of \mathcal{S}_i and of \mathcal{S}_j (T5-2b, c), and hence an [*L*]implicate in *S* both of \mathcal{S}_i and of \mathcal{S}_j .

([I] T33[34]-8e). I, for (a). Therefore, if \mathcal{S}_i is true in S , \mathcal{S}_k is true ([I] T9-10); thus Dj_1 and Dj_2 are generally satisfied (T11-11). And if \mathcal{S}_i is true, \mathcal{S}_k is true; thus Dj_3 is generally satisfied. — II, for a $[b]$. Since $\mathcal{S}_i \rightarrow [L] \mathcal{S}_k$ in S , $-R_i + R_k$ contains rs $[is V_a]$ ([I] T20-28[10]). Likewise, $-R_j + R_k$ contains rs $[is V_a]$. Hence, also $-R_i + (-R_j) + R_k$, $-R_i + R_j + R_k$, and $R_i + (-R_j) + R_k$ contain rs $[are V_a]$. Therefore, $disc$ has the $[L-]$ characteristic value T for the first, second, and third distribution in S (D11-16). Since the first, second, and third values in the characteristic of disjunction are T (see column (5) in the table in § 10), $disc$ $[L-]$ satisfies generally Dj_1 , 2, and 3 (D11-17).

T15-5. Let us suppose that K and S fulfill the following conditions (without asserting that this is possible):

- A. K contains PC_1 .
- B. S is a true interpretation for K .
- C. neg_C in K violates the rule N_1 of NTT at least once in S , say with respect to \mathcal{S}_1 .

Then the following holds:

- a. Both \mathcal{S}_1 and $neg_C(\mathcal{S}_1)$ are true.
- b. Every sentence of K is true in S .
- c. neg_C always violates N_1 .
- d. N_2 is not violated by neg_C , nor Dj_2 , 3, and 4 by $disc$; but these rules have no instances of application.

Proof. I. a. N_1 (§ 10) applied to neg_C says that, if \mathcal{S}_i is true in S , $neg_C(\mathcal{S}_i)$ is false. Hence, the violation of N_1 with respect to \mathcal{S}_1 (C) means that both \mathcal{S}_1 and $neg_C(\mathcal{S}_1)$ are true. — b. $\{\mathcal{S}_1, neg_C(\mathcal{S}_1)\}$ is true ((a), [I] D9-1). Every sentence of K is a C-implicate of this class in K (T5-21) and hence an implicate of it in S (B) and hence also true in S . — c. For every closed \mathcal{S}_i in K , both \mathcal{S}_i and $neg_C(\mathcal{S}_i)$ are true in S (b); hence N_1 is always violated. — d. From (b). — II. a. Since neg_C violates N_1 with respect to \mathcal{S}_1 (D11-15), \mathcal{S}_1 has the first distribution, i.e. T (D10-2), and $neg_C(\mathcal{S}_1)$ does not have the first value in the characteristic for negation, which is F; thus it too has T. — b. From (a) (as in I). — c. From (b). — d. From (b), T12-16a(4), T12-17a(4).

T15-6 (Corollary). If K contains PC_1 and S is a true

interpretation for K and at least one sentence of K is false in S , then neg_C in K generally satisfies the rule N1 in S . (From T5b, T11-11.)

T15-7. Let us suppose that K and S fulfill the following conditions (without asserting that this is possible):

A and **B**, as in T5.

C. neg_C in K violates the rule N2 of NTT at least once in S , say with respect to \mathcal{S}_1 ; let \mathcal{S}_3 be $\text{dis}_C(\mathcal{S}_1, \text{neg}_C(\mathcal{S}_1))$.

Then the following holds:

- a. Both \mathcal{S}_1 and $\text{neg}_C(\mathcal{S}_1)$ are false.
- b. neg_C in K generally satisfies N1 in S .
- c. \mathcal{S}_3 is true.
- d. dis_C in K violates Dj₄ with respect to \mathcal{S}_1 , $\text{neg}_C(\mathcal{S}_1)$.
- e. $\text{neg}_C(\mathcal{S}_3)$ is false.
- f. $\text{neg}_C(\text{neg}_C(\mathcal{S}_3))$ is true.
- g. neg_C satisfies N2 with respect to $\text{neg}_C(\mathcal{S}_3)$.
- h. neg_C in K is non-extensional in S .
- i. If K , moreover, fulfills the conditions (B) and (C) in T6-10 and contains another sign of negation_C, neg'_C , then this sign too violates N2 and is non-extensional in S .

Proof. a. From (C), in analogy to T5a. — b. From (a), T6. — c. \mathcal{S}_3 is C-true in K (T5-1a), and hence true in S (B). — d. From (a), (c). — e. From (b), (c), (II: T11-11). — f. $\text{neg}_C(\text{neg}_C(\mathcal{S}_3))$ is C-equivalent to \mathcal{S}_3 in K (T5-3a), hence equivalent to it in S (B), and hence true (c). — g. From (e), (f). — h. From (g), (C), T12-6. — i. For any closed \mathcal{S}_i , $\text{neg}_C(\mathcal{S}_i)$ and $\text{neg}'_C(\mathcal{S}_i)$ are C-equivalent in K (T8-9a) and hence equivalent in S (B). Therefore neg'_C satisfies and violates N2 with respect to the same sentences as neg_C .

T15-8. Let us suppose that K and S fulfill the following conditions (without asserting that this is possible):

A and **B**, as in T5.

C. disc in K violates the rule Dj_4 of NTT at least once in S , say with respect to $\mathcal{S}_1, \mathcal{S}_2$.

Then the following assertions (a) to (g) hold:

- a. Both \mathcal{S}_1 and \mathcal{S}_2 are false, $\text{disc}(\mathcal{S}_1, \mathcal{S}_2)$ is true.
- b. \mathcal{S}_1 and \mathcal{S}_2 are different.
- c. neg_C in K generally satisfies N_1 in S .
- d. $\text{neg}_C(\mathcal{S}_1)$ is false; hence, neg_C violates N_2 with respect to \mathcal{S}_1 ; the same holds for \mathcal{S}_2 .
- e. $\text{disc}(\text{neg}_C(\mathcal{S}_1), \mathcal{S}_2)$ is false; hence, disc satisfies Dj_4 with respect to $\text{neg}_C(\mathcal{S}_1), \mathcal{S}_2$; the same holds for $\mathcal{S}_1, \text{neg}_C(\mathcal{S}_2)$.
- f. disc in K is non-extensional in S .
- g. neg_C in K is non-extensional in S .

If K , moreover, fulfills the conditions (B) and (C) in T6-10, then, in addition, the following assertions (k) to (n) hold:

- k. If K contains another sign of disjunction_C, say disc' , then this sign, too, violates Dj_4 and is non-extensional.
- l. If K contains another sign of negation_C, say neg'_C , this sign, too, violates N_2 and is non-extensional.
- m. Every sentence \mathcal{S}_i which is a C-implicate in K both of \mathcal{S}_1 and of \mathcal{S}_2 is true in S .
- n. \mathcal{S}_2 is not a C-implicate of \mathcal{S}_1 in K ; nor \mathcal{S}_1 of \mathcal{S}_2 .

Proof. a. From (C), in analogy to T5a. — b. If \mathcal{S}_2 were \mathcal{S}_1 , then \mathcal{S}_1 , being a C-implicate of $\text{disc}(\mathcal{S}_1, \mathcal{S}_1)$ in K (T5-2a), would be a C-implicate of $\text{disc}(\mathcal{S}_1, \mathcal{S}_2)$ in K and hence an implicate of this sentence in S (B), and hence true in S like this sentence (a). But \mathcal{S}_1 is not true (a). Therefore \mathcal{S}_2 must be different from \mathcal{S}_1 . — c. From (a), T6. — d. \mathcal{S}_2 is a C-implicate of $\{\text{disc}(\mathcal{S}_1, \mathcal{S}_2), \text{neg}_C(\mathcal{S}_1)\}$ in K (T5-2e) and hence an implicate of this class in S (B). $\text{disc}(\mathcal{S}_1, \mathcal{S}_2)$ is true (a); if now $\text{neg}_C(\mathcal{S}_1)$ were true, the class mentioned would be true and hence \mathcal{S}_2 too. But this is not the case (a). Therefore $\text{neg}_C(\mathcal{S}_1)$ cannot be true and must be false. Since \mathcal{S}_1 is false (a), N_2 is violated. The reasoning for $\text{neg}_C(\mathcal{S}_2)$ is analogous. — e. \mathcal{S}_2 is a C-implicate of

$\{\text{disc}(\mathcal{S}_1, \mathcal{S}_2), \text{disc}(\text{neg}_C(\mathcal{S}_1), \mathcal{S}_2)\}$ in K (T5-2h), and hence an implicate of this class in S (B). The first element of the class is true (a); if now the second were true, \mathcal{S}_2 would be true; but it is not (a). Therefore, $\text{disc}(\text{neg}_C(\mathcal{S}_1), \mathcal{S}_2)$ must be false. Hence, Dj_4 is satisfied in this case (d, a). The reasoning for $\text{disc}(\mathcal{S}_1, \text{neg}_C(\mathcal{S}_2))$ is analogous. — f. From (e), (C), T12-6. — g. From T7h because N_2 is violated (d).

k. From (C), T7-4a (the proof is analogous to that of T7i). — l. From (d), T8-9a. — m. If \mathcal{S}_i is a C-implicate both of \mathcal{S}_1 and of \mathcal{S}_2 , it is a C-implicate of $\text{disc}(\mathcal{S}_1, \mathcal{S}_2)$ in K (T7-2b) and hence an implicate of this sentence in S (B) and hence true because $\text{disc}(\mathcal{S}_1, \mathcal{S}_2)$ is true (a). — n. If \mathcal{S}_2 were a C-implicate of \mathcal{S}_1 in K it would be true in S (m). But it is not true (a). Analogously for \mathcal{S}_1 .

T15-9. Let K contain PC_1 .

- a. If neg_C in K has a normal interpretation in S , then disc_C likewise.
- b. If disc_C in K has a normal interpretation in S and at least one sentence of K is false in S , then neg_C also has a normal interpretation in S .

Proof. a. Let neg_C have a normal interpretation in S . Then it is a sign of negation in S (D1) and hence does not violate N_2 with respect to any sentence (T11-12a, T11-11). Therefore, disc_C does not violate Dj_4 in any case (T8d) and hence generally satisfies Dj_4 (T11-11). Further, disc_C generally satisfies Dj_1 to 3 (T4). Hence, it is a sign of disjunction in S (T11-12a) and has a normal interpretation (D1). — b. Let the conditions be fulfilled. Then (in analogy to (a)) disc_C does not violate Dj_4 in any case (D1, T11-12a, T11-11). Therefore, neg_C generally satisfies N_2 (T7d, T11-11), and also N_1 (T6). Hence, it is a sign of negation (T11-12a).

§ 16. Non-Normal Interpretations in General

The possibilities of non-normal (true) interpretations for all singulary and binary connectives in PC are examined with the help of NTT (see table). It is found that the sign of conjunction_C and some other less important connectives always (i.e. in any true interpretation for a calculus K containing PC_1^D) have a normal interpretation (Tr). If the sign of negation_C has a normal interpretation, then every other connective has too (T₃). We distinguish two kinds of non-normal interpretations; in the first kind, every sentence (in K) is true (in S); in the second kind, at least one is false. For any case of the first kind, the following holds (T₆; columns (5) to (7) of the table): the singulary connectives nos. 1 and 2, and the binary nos. 1 through 8 have a normal interpretation, but the others have not; all connectives are extensional. For any case of the second kind, the following holds (T₇, columns (8) to (10) of the table): the singulary connectives nos. 1, 2, and 4 and the binary nos. 1, 4, 6, 8, and 16 have a normal interpretation; the others have a non-normal interpretation and are non-extensional.

So far, we have discussed the question of interpretations only for signs of negation_C and disjunction_C. Now we shall examine other connectives in PC. In column (2), the table that follows lists again the connections_C in a calculus K containing PC, as they were previously listed in the table in § 3. Column (3) here repeats column (5) of the previous table; it gives expressions for the connections_C in PC_1 , which are taken as definientia for the defined signs in PC_1^D (D₃₋₆) on the basis of neg_C and dis_C. Column (4) repeats column (5) of the table in § 10; it gives the characteristics for the corresponding connections on the basis of the rules of NTT. Columns (5) to (10) give a survey of some of the results concerning non-normal interpretations, as stated in the subsequent theorems, especially T₆ and 7.

NON-NORMAL INTERPRETATIONS OF PROPOSITIONAL CONNECTIVES OF PC

(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Name of Connections in K	Definiens for the Sign in K	Charac- teristic for the Sign in NTT	Non-Normal Interpretations					
			first kind (T6) (all sentences true)			second kind (T7) (not all sentences true)		
			normal or not	which rule vio- lated	exten- sional or not	normal or not	which rule vio- lated	exten- sional or not
<i>I. The four singular connections_c</i>								
tautology _c	$\mathcal{S}_i V \sim \mathcal{S}_i$	TT	n	-	e	n	-	e
(identity) _c	\mathcal{S}_i	TF	n	-	e	n	-	e
negation _c	$\sim \mathcal{S}_i$	FT	-	I	e	-	2	-
contradiction _c	$\sim (\mathcal{S}_i V \sim \mathcal{S}_i)$	FF	-	I	e	n	-	e
<i>II. The sixteen binary connections_c</i>								
tautology _c	$\mathcal{S}_i V \sim \mathcal{S}_i$	TTTT	n	-	e	n	-	e
disjunction _c	$\mathcal{S}_i V \mathcal{S}_i$	TTTF	n	-	e	-	4	-
(inverse impli- cation) _c	$\mathcal{S}_i V \sim \mathcal{S}_i$	TTFT	n	-	e	-	4	-
(first compo- nent)	\mathcal{S}_i	TTF	n	-	e	n	-	e
implication _c	$\sim \mathcal{S}_i V \mathcal{S}_i$	TFTT	n	-	e	-	4	-
(second compo- nent)	\mathcal{S}_i	TFTF	n	-	e	n	-	e
equivalence _c	$\sim (\sim \mathcal{S}_i V \sim \mathcal{S}_i)$ $V \sim (\mathcal{S}_i V \mathcal{S}_i)$	TFFT	n	-	e	-	4	-
conjunction _c	$\sim (\sim \mathcal{S}_i V \sim \mathcal{S}_i)$	TFFF	n	-	e	n	-	e
exclusion _c	$\sim \mathcal{S}_i V \sim \mathcal{S}_i$	FTTT	-	I	e	-	4	-
(non-equa- lence) _c	$\sim (\sim \mathcal{S}_i V \mathcal{S}_i)$	FTTF	-	I	e	-	4	-
(negation _c of second)	$V \sim (\mathcal{S}_i V \sim \mathcal{S}_i)$	FTFT	-	I	e	-	4	-
(first alone)	$\sim (\sim \mathcal{S}_i V \mathcal{S}_i)$	FTFF	-	I	e	-	2	-
(negation _c of first)	$\sim \mathcal{S}_i$	FFTT	-	I	e	-	4	-
(second alone)	$\sim (\mathcal{S}_i V \sim \mathcal{S}_i)$	FFTF	-	I	e	-	3	-
bi-negation _c	$\sim (\mathcal{S}_i V \mathcal{S}_i)$	FFFT	-	I	e	-	4	-
contradiction _c	$\sim (\mathcal{S}_i V \sim \mathcal{S}_i)$	FFFF	-	I	e	n	-	e

Conjunction_C and disjunction_C are often regarded as playing completely symmetrical roles in PC (the so-called duality). However, we now find (T₁) that for con_C, in contradistinction to dis_C, only a normal interpretation is possible. Thus the supposed symmetry, although it is perfect within NTT, holds in PC only to a certain extent. The reason is that the rules of deduction in any of the ordinary forms of PC, in contradistinction to NTT, are in a certain sense incomplete with respect to disjunction_C but not to conjunction_C. This will become clearer later.

+T16-1a [b]. Let K contain PC_1^D , and S be an $[L]$ -true interpretation for K . Then each of the following connectives in K (see tables here and in § 3) is a sign for the corresponding connection_[L] in S and hence has an $[L]$ -normal interpretation in S :

1. Two singular connectives: cb_1 and cb_2 .
2. Four binary connectives: ct_r for $r = 1$ (tautology_C), 4 (first component), 6 (second component), 8 (conjunction_C).

Proof for a [b]. 1. Let \mathcal{S}_i be any closed sentence in K , and \mathcal{S}_k be $cb_1(\mathcal{S}_i)$. On the basis of the definition-rule for cb_1 (see D3-6), \mathcal{S}_k is C-equivalent in K to $disc(\mathcal{S}_i, neg_C(\mathcal{S}_i))$ (see column (3) of the table, line I₁), and hence C-true in K like the latter sentence (T_{5-1a}) and hence $[L]$ -true in S . Therefore, R_k contains rs [is V_*], and hence likewise $\neg R_i + R_k$ and $R_i + R_k$. Therefore, cb_1 has T as the $[L]$ -characteristic value both for the first and the second distribution (D₁₁₋₁₆), and hence has TT as its $[L]$ -characteristic (D₁₁₋₂₁) and is a sign for $_{[L]}Conn_1^1$ (D₁₁₋₂₃) and has an $[L]$ -normal interpretation (D₁₅₋₁). — Let \mathcal{S}_i be closed and \mathcal{S}_k be $cb_2(\mathcal{S}_i)$. \mathcal{S}_k is C-equivalent in K (D3-6) and hence $[L]$ -equivalent in S to \mathcal{S}_i . Hence, both $\neg R_i + R_k$ and $R_i + (\neg R_k)$ contain rs [are V_*] (T₁₁₋₆₍₂₎). Therefore, cb_2 has T as the $[L]$ -characteristic value for $t = 1$ and F for $t = 2$ (D₁₁₋₁₆), and hence has TF as its $[L]$ -characteristic (D₁₁₋₂₁) and is a sign for $_{[L]}Conn_2^1$ (D₁₁₋₂₃) and has an $[L]$ -normal interpretation (D₁₅₋₁). — 2. The proof for ct_1 is analogous to that for cb_1 . The proofs for ct_4 and ct_6 are analogous to that for cb_2 . — Proof for $ct_8 (= con_C)$. Let

\mathcal{S}_i and \mathcal{S}_j be closed, and \mathcal{S}_k be $\text{conc}(\mathcal{S}_i, \mathcal{S}_j)$. \mathcal{S}_k and $\{\mathcal{S}_i, \mathcal{S}_j\}$ are C-equivalent in K (T5-3b) and hence $[L]$ -equivalent in S . Therefore, conc is a sign of conjunction $_{[L]}$ in S (T13-14).

+T16-2a [b]. If K contains PC_1^D , and neg_C and dis_C in K have an $[L]$ -normal interpretation in S , then every other connective of PC_1^D in K also has an $[L]$ -normal interpretation in S .

Proof for $\text{ct}_3 (= \text{imp}_C)$; the proofs for the other connectives are similar. Let \mathcal{S}_i and \mathcal{S}_j be any closed sentences in K . Let \mathcal{S}_k be $\text{neg}_C(\mathcal{S}_i)$, \mathcal{S}'_k be $\text{dis}_C(\mathcal{S}_i, \mathcal{S}_j)$, \mathcal{S}_p be $\text{imp}_C(\mathcal{S}_i, \mathcal{S}_j)$. (For 'I' and 'II', see remark preceding T15-4.) — I, for (a). \mathcal{S}'_k is true for the first, third, and fourth distribution, false for the second (this follows easily from the rules N1 and 2, D11 to 4). The same holds for \mathcal{S}_p , since \mathcal{S}_p and \mathcal{S}'_k are C-equivalent in K according to the definition of imp_C (D3-6), and hence equivalent in S , which is a true interpretation for K (D15-1a). Therefore, imp_C has the characteristic TFFT and hence is a sign of implication in S and has a normal interpretation in S (D15-1a). — II, for a [b]. Each of the following classes contains rs [is V_s] (T13-5, T13-10(1) to (4)): $R_i + R_k$ (k_1), $-R_i + (-R_k)$ (k_2), $-R_k + (-R_j) + R'_k$ (k_3), $-R_k + R_j + R'_k$ (k_4), $R_k + (-R_j) + R'_k$ (k_5), $R_k + R_j + (-R'_k)$ (k_6). \mathcal{S}_p and \mathcal{S}'_k are $[L]$ -equivalent (see I); hence $R_p + (-R'_k)$ (k_7) and $-R_p + R'_k$ (k_8) contain rs [are V_s] (T11-6(2)). Therefore each of the following classes also contains rs [is V_s]: $-R_i + (-R_j) + R_p$ ($= k_2 + k_5 + k_7$), $-R_i + R_j + (-R_p)$ ($= k_2 + k_6 + k_8$), $R_i + (-R_j) + R_p$ ($= k_1 + k_3 + k_7$), $R_i + R_j + R_p$ ($= k_1 + k_4 + k_7$). Hence, imp_C has the $[L]$ -characteristic value T for $t = 1$ (D11-16), F for $t = 2$, T for $t = 3$, T for $t = 4$. Thus it has the $[L]$ -characteristic TFFT (D11-21) and hence is a sign for $_{[L]}\text{Corr}_5^2 (= \text{implication}_{[L]})$ and has an $[L]$ -normal interpretation in S (D15-1).

+T16-3. If K contains PC_1^D , and neg_C in K has a normal interpretation in S , then every other connective of PC_1^D in K also has a normal interpretation in S . (From T15-9a, T2a.)

+T16-4 (Corollary). If K contains PC_1^D , and dis_C in K has a normal interpretation in S , and at least one sentence of K is false in S , then every other connective of PC_1^D in K also has a normal interpretation in S . (From T15-9b, T3.)

On the basis of the previous discussion of non-normal interpretations for neg_c and dis_c we can now characterize in general the kinds of non-normal interpretations of connectives. There we found two kinds of cases where the rules of NTT for negation or disjunction are violated: there is either a violation of N_1 alone (T15-5) or a simultaneous violation of N_2 and Dj_4 (T15-7 and 8). In a case of the first kind all sentences of K are true in S , while in a case of the second kind at least one is false. This difference yields a convenient way of defining the two kinds. Theorems T6 and 7, below, state some properties of cases of the two kinds without asserting the existence of such cases. These two kinds exhaust all possibilities for non-normal interpretations for any connective of PC in K . In columns (5) to (10) of the table, some of the results stated in T6 and 7 are listed.

+T16-6. Let K and S fulfill the following conditions (*non-normal interpretation of the first kind*):

- A. K contains PC_1^D .
- B. S is a true interpretation for K .
- C. All sentences of K are true in S .

Then the following holds:

- a. The following ten connectives in K do not have a normal interpretation in S : $c b_q$ for $q = 3$ and 4 ; $c c_r$ for $r = 9$ to 16 .
- b. The other connectives in K have a normal interpretation in S : $c b_q$ for $q = 1$ and 2 ; $c c_r$ for $r = 1$ to 8 .
- c. Every connective in K is extensional in S .
(a. From T12-16a(7), T12-17a(7). b. From T12-16a(6), T12-17a(6). c. From T12-16a(8), T12-17a(8).)

+T16-7. Let K and S fulfill the following conditions (*non-normal interpretation of the second kind*):

- A, B, as in T6.

- C. At least one sentence of K is false in S .
- D. At least one of the connectives of PC in K has not a normal interpretation in S .

Then the following holds:

- a. An infinite number of sentences of K are false in S .
- b. An infinite number of sentences of K are true in S .
- c. neg_C in K violates N_2 , but generally satisfies N_1 .
- d. neg_C in K is non-extensional.
- e. The three other singulary connectives in K ($\text{c}b_q$ for $q = 1, 2, 4$) have a normal interpretation in S .
- f. The following eleven binary connectives cc_r in K violate a rule for Conn_r^2 in NTT and hence do not have a normal interpretation in S : $r = 2, 3, 5, 7, 9, 10, 11, 12, 13, 14, 15$. For $r = 2, 3, 5, 7, 9, 10, 11, 13, 15$, at least the fourth rule is violated; for $r = 12$, the second; for $r = 14$, the third.
- g. The connectives of K mentioned in (f) are non-extensional in S .
- h. The five other binary connectives in K (cc_r for $r = 1, 4, 6, 8, 16$) have a normal interpretation in S .

Proof. Let the conditions (A) to (D) be fulfilled. Then the following holds. — 1. neg_C generally satisfies N_1 (T15-6). — 2. neg_C in K does not have a normal interpretation in S (C, T3). — 3. neg_C violates N_2 at least once (1, 2), say with respect to \mathcal{S}_1 . — 4. \mathcal{S}_1 and $\text{neg}_C(\mathcal{S}_1)$ are false in S ((3), T15-7a). Let \mathcal{S}_3 be $\text{disc}(\mathcal{S}_1, \text{neg}_C(\mathcal{S}_1))$, and \mathcal{S}_4 be $\text{disc}(\mathcal{S}_3, \text{neg}_C(\mathcal{S}_3))$. — 5. \mathcal{S}_3 is true (T15-7c). — 6. $\text{neg}_C(\mathcal{S}_3)$ is false (T15-7e). — 7. \mathcal{S}_4 is C-true in K (T5-1a) and hence true in S (B). — 8. $\text{neg}_C(\mathcal{S}_4)$ is false in S (7, 1). — 9. An infinite number of sentences in K are C-equivalent to \mathcal{S}_1 in K (e.g. \mathcal{S}_1 with neg_C added $2n$ times),

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hence equivalent to \mathcal{S}_1 in S (B), hence false in S (4). This is (a). — 10. An infinite number of sentences are C-equivalent to \mathcal{S}_3 in K and hence true ((B), (5)). This is (b). — 11. (c) from (3), (1). — 12. (d) from (3), T15-7h. — 13. For any closed sentence \mathcal{S}_1 in K , $\text{disc}(\mathcal{S}_1, \text{neg}_C(\mathcal{S}_1))$ is C-true in K (T5-1a) and hence true in S (B). $\text{neg}_C(\text{disc}(\mathcal{S}_1, \text{neg}_C(\mathcal{S}_1)))$ is false in S (1). $\text{cb}_4(\mathcal{S}_1)$ is C-equivalent in K and hence equivalent in S to the sentence just mentioned (see column (3) of the table, line I4) and hence is also false in S . Therefore, cb_4 has the characteristic value F both for $t = 1$ and $t = 2$, and hence the characteristic FF, and hence is a sign for Conn_4^1 in S (D11-23a) and has a normal interpretation in S (D15-1a). — 14. (e) from T1a(1) and (13). — 15. Let $\text{c}\mathcal{S}_r$ ($r = 1$ to 16) be $\text{c}\mathcal{S}_r(\mathcal{S}_1, \text{neg}_C(\mathcal{S}_1))$. $\text{c}\mathcal{S}_r$ is C-equivalent in K and hence equivalent in S to the sentence given in column (3) of the table, but with \mathcal{S}_1 instead of \mathcal{S}_i and $\text{neg}_C(\mathcal{S}_1)$ instead of \mathcal{S}_j (' \sim ' and ' \vee ' are neg_C and disc in K). Each of these sentences, in turn, can easily be transformed (chiefly by virtue of T5-1 and 3) into a certain other sentence which is C-equivalent to it in K and hence equivalent to it in S . In this way we find (line II3 of the table) that $\text{c}\mathcal{S}_3$ is equivalent in S to $\text{disc}(\mathcal{S}_1, \text{neg}_C(\text{neg}_C(\mathcal{S}_1)))$ and further to \mathcal{S}_1 , and hence is false in S (4); $\text{c}\mathcal{S}_6$ is equivalent to $\text{neg}_C(\mathcal{S}_1)$ and hence false (4); $\text{c}\mathcal{S}_7$ is equivalent to $\text{neg}_C(\mathcal{S}_3)$ and hence false (6); $\text{c}\mathcal{S}_{10}$ is equivalent to \mathcal{S}_3 and hence true (5); $\text{c}\mathcal{S}_{11}$ is equivalent to \mathcal{S}_1 and hence false (4); $\text{c}\mathcal{S}_{13}$ is equivalent to $\text{neg}_C(\mathcal{S}_1)$ and hence false (4); $\text{c}\mathcal{S}_{15}$ is equivalent to $\text{neg}_C(\mathcal{S}_3)$ and hence false (6). \mathcal{S}_1 and $\text{neg}_C(\mathcal{S}_1)$ are both false (4) and hence have the fourth distribution of values (D10-2). Therefore the fourth characteristic value of $\text{c}\mathcal{S}_3$ is F (T11-10), since $\text{c}\mathcal{S}_3$ is false; the same holds for $\text{c}\mathcal{S}_r$ with $r = 5, 7, 11, 13, 15$; but that for $\text{c}\mathcal{S}_{10}$ is T. — 16. The fourth value in the characteristic for Conn_2^2 for $r = 3, 5, 7, 11, 13, 15$ is T, that for Conn_{10}^2 is F. — 17. For $r = 3, 5, 7, 10, 11, 13, 15$, $\text{c}\mathcal{S}_r$ violates the fourth rule for Conn_2^2 in S . — 18. $\text{c}\mathcal{S}_9(\mathcal{S}_1, \mathcal{S}_1)$ is equivalent to $\text{neg}_C(\mathcal{S}_1)$ (in analogy to (13)) and hence false (4). — 19. \mathcal{S}_1 is false (4); hence $\text{c}\mathcal{S}_9$ (18) violates the fourth rule for Conn_9^2 (in analogy to (15), (16), (17)). — 20. Let us analyze the sentences $\text{c}\mathcal{S}_r(\text{neg}_C(\mathcal{S}_3), \text{neg}_C(\mathcal{S}_3))$, which we call $\text{c}\mathcal{S}'_r$. $\text{c}\mathcal{S}'_3$ is C-equivalent in K (in analogy to (15)) and hence equivalent in S to \mathcal{S}_4 and hence is true in S (7); the same holds for $\text{c}\mathcal{S}'_5$ and $\text{c}\mathcal{S}'_7$; $\text{c}\mathcal{S}'_9$ is equivalent to \mathcal{S}_3 and hence true (5); the same holds for $\text{c}\mathcal{S}'_{11}$, $\text{c}\mathcal{S}'_{13}$, and $\text{c}\mathcal{S}'_{15}$; $\text{c}\mathcal{S}'_{10}$ is equivalent to $\text{neg}_C(\mathcal{S}_4)$ and hence false (8). — 21. Since both components in $\text{c}\mathcal{S}_r$ are false (6), they have the fourth value distribution. The fourth value in the characteristic for Conn_2^2 for $r = 3, 5, 7, 9, 11, 13, 15$ is T,

that for $r = 10$ is F. Thus, for $r = 3, 5, 7, 9, 10, 11, 13, 15$, cc_r satisfies the fourth rule for Conn_2^2 with respect to the components mentioned. — 22. $cc_{12}(\mathcal{S}_3, \mathcal{S}_1)$ is equivalent to $\text{neg}_C(\text{disc}(\text{neg}_C(\mathcal{S}_3), \mathcal{S}_1))$ and further to $\text{neg}_C(\mathcal{S}_1)$ and hence is false (4). — 23. Since \mathcal{S}_3 is true (5) and \mathcal{S}_1 is false (4), they have the second distribution. The second value in the characteristic for Conn_{12}^2 is T. Hence, cc_{12} (22) violates the second rule for Conn_{12}^2 . — 24. $cc_{12}(\mathcal{S}_3, \text{neg}_C(\mathcal{S}_3))$ is equivalent to \mathcal{S}_3 and hence is true (5). — 25. The components mentioned (24) have the second distribution. The second value in the characteristic for Conn_{12}^2 is T. Hence, cc_{12} satisfies the second rule for Conn_{12}^2 with respect to the components mentioned. — 26. $cc_{14}(\mathcal{S}_1, \mathcal{S}_3)$ is equivalent to $\text{neg}_C(\mathcal{S}_1)$ and hence false (4). — 27. The third value in the characteristic for Conn_{14}^2 is T. Hence, cc_{14} violates the third rule for Conn_{14}^2 (26). — 28. $cc_{14}(\text{neg}_C(\mathcal{S}_3), \mathcal{S}_3)$ is equivalent to \mathcal{S}_3 and hence true. (5). — 29. The third value for Conn_{14}^2 is T. Hence, cc_{14} satisfies the third rule for Conn_{14}^2 in this case (28). — 30. For each of the connectives cc_r for $r = 3, 5, 7, 9, 10, 11, 13, 15$ in K , the fourth rule for the corresponding connection Conn_2^2 in NTT is sometimes violated (17, 19), sometimes satisfied (21). For cc_{12} , the second rule is sometimes violated (23), sometimes satisfied (25). For cc_{14} , the third rule is sometimes violated (27), sometimes satisfied (29). — 31. (f) from T15-7d (for $r = 2$) and (30). — 32. (g): for $r = 2$, from T15-7d and T15-8f; for the rest from (30) and T12-6. — 33. cc_{18} in K has a normal interpretation in S ; the proof is analogous to that for cb_4 (13). — 34. (h) from Tra (2) and (33).

+T16-8. If K contains PC_1^D and one of the connectives cc_r for $r = 9$ through 15 has a normal interpretation in S , then every other connective of PC_1^D in K also has a normal interpretation in S .

Proof. If one of the other connectives had a non-normal interpretation, then it would be a case either of the first or the second kind. In both cases all connectives mentioned would have a non-normal interpretation (T6a, T7f).

neg_C (T3) and the seven binary connectives mentioned in T8 are the only connectives of PC_1^D in K having the property stated in T8. Every other connective has a normal interpretation in at least one of the two kinds of non-normal

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interpretations (T6 and 7; compare columns (5) and (8) in the table).

§ 17. Examples of Non-Normal Interpretations

In § 16 two kinds of non-normal interpretations for the connectives in PC were studied without showing that these kinds are non-empty. This is shown here by the construction of examples for true and, moreover, L-true interpretations of both kinds.

The two kinds of non-normal interpretations for the connectives of PC which were referred to in T16-6 and 7 exhaust all possibilities of non-normal interpretations; this is seen from the conditions (C) in the two theorems. Thus there are at most these two kinds. But so far we have not seen whether there really are non-normal interpretations of these kinds. This will now be shown by examples.

For the following examples we shall take a calculus K and two semantical systems S and S' which fulfill the following conditions:

A. K contains n propositional constants, say ' A_1 ', ' A_2 ', ... ' A_n '.

B. K contains PC_1 or PC_1^D .

C. K contains no other sentences than the molecular sentences constructed out of the propositional constants with the help of the connectives of PC (hence no variables, and only closed sentences).

D. K contains no other rules of deduction than those of PC (hence no rule of refutation; all rules of inference are extensible).

E. The sentences of S are those of K . Hence, S is an interpretation for K .

F. S' contains $n + 2$ atomic sentences, say ' A'_1 ', ' A'_2 ', ... ' A'_n ', ' A'_{n+1} ' (\mathfrak{C}_1), ' A'_{n+2} ' (\mathfrak{C}_2) such that the following

holds: **a.** Each of the atomic sentences is L-independent of the rest (they may e.g. be full sentences of a predicate for $n + 2$ different objects). — **b.** Hence, all are factual. — **c.** \mathfrak{S}_1 is true, and hence F-true. — **d.** \mathfrak{S}_2 is false, and hence F-false. — **e.** Let \mathfrak{S}_3 be an L-true sentence in S' (e.g. $\text{dis}(\mathfrak{S}_1, \text{neg}(\mathfrak{S}_1))$, compare (G)). [The truth-rules for the atomic sentences in S' are supposed to be given so as to fulfill (F); in any other respect they may be chosen arbitrarily. We do not give them because their details beyond (F) are irrelevant for the nature of the interpretations in the examples.]

G. S' contains NTT. Hence the connectives of NTT in S' are signs for the connections_L.

H. If \mathfrak{S}_i is a sentence of K and hence of S , then we designate by ' \mathfrak{S}'_i ' the corresponding sentence in S' , that is to say, the sentence constructed out of \mathfrak{S}_i by replacing each propositional constant that occurs, say ' A_k ' ($k = 1$ to n), by the corresponding atomic sentence in S' , ' A'_k ', and replacing each connective that occurs by the corresponding connective in S' . [Hence, if \mathfrak{S}_i is $\text{neg}_C(\mathfrak{S}_j)$, \mathfrak{S}'_i is $\text{neg}_L(\mathfrak{S}'_j)$, and if \mathfrak{S}_i is $\text{dis}_C(\mathfrak{S}_j, \mathfrak{S}_k)$, \mathfrak{S}'_i is $\text{dis}_L(\mathfrak{S}'_j, \mathfrak{S}'_k)$.] If \mathfrak{R}_i is a sentential class in K and S , then we designate by ' \mathfrak{R}'_i ' the class of the corresponding sentences in S' .

In the following examples, the systems K and S' remain always the same. S differs from example to example. In each case we shall describe the system S by stating a translation of the sentences of S into some sentences of S' . The translation is meant in this way: the truth-rules in S state for the sentence \mathfrak{S}_i the same truth-condition as the rules in S' state for the sentence \mathfrak{S}_j , into which \mathfrak{S}_i is translated. Therefore, if any radical or L-concept holds for \mathfrak{S}_j in S' , then the same concept holds for \mathfrak{S}_i in S . [If we use the concept of L-equivalence also for sentences in different systems (compare remark at the end of [I] § 16), then \mathfrak{S}_i and \mathfrak{S}_j are L-equivalent.] If we were to translate every sentence \mathfrak{S}_i in S into

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the corresponding sentence \mathfrak{S}_i' in S' , then S would be an L-true interpretation for K , and each connective in K would have an L-normal interpretation in S . Therefore, in order to construct non-normal interpretations for the connectives in K , other translations have to be made. In each of the examples it will be shown that S is an L-true interpretation for K such that at least one connective in K has a non-normal interpretation in S . The first two examples of interpretations are rather trivial, but they suffice to show in a simple way that both kinds of non-normal interpretations previously explained are not empty.

First example: an L-true, non-normal interpretation of the first kind. We translate every sentence in S into \mathfrak{S}_3 ($F(e)$). Then the following holds: a. Every sentence in S is L-true. b. S is an L-true interpretation for K . c. neg_C in K violates N_1 in S . d. S is a non-normal interpretation of the first kind.

Proof. a. Every sentence in S is L-equivalent to an L-true sentence and hence L-true. — b. For every \mathfrak{X}_i and \mathfrak{X}_j in S , \mathfrak{X}_i and \mathfrak{X}_j are L-true (a), and hence $\mathfrak{X}_i \rightarrow \mathfrak{X}_j$. Thus condition (a) in [I] D34-1 is fulfilled. Condition (b) in the same definition is always fulfilled because of (D). Hence, S is an L-true interpretation for K . — c. For any \mathfrak{S}_i in K , both \mathfrak{S}_i and $\text{neg}_C(\mathfrak{S}_i)$ are true (a). — d. From T16-6.

Second example: an L-true, non-normal interpretation of the second kind. A sentence \mathfrak{S}_m of S is translated, if it is C-true in K , into \mathfrak{S}_3 (which is L-true, see ($F(e)$)), otherwise into $\text{neg}(\mathfrak{S}_3)$ (which is L-false). Then the following holds: a. S is an L-true interpretation for K . b. neg_C in K violates N_2 in S . c. S is a non-normal interpretation of the second kind.

Proof. a. Let the conditions be fulfilled, and \mathfrak{S}_i be a direct C-implicate of \mathfrak{R}_i in K . If \mathfrak{R}_i is C-true in K , \mathfrak{S}_i is C-true in K and is hence translated into an L-true sentence in S' ; therefore, in this case, \mathfrak{S}_i is L-true in S , and hence an L-implicate of \mathfrak{R}_i in S . If, on the other hand,

\mathfrak{R}_i is not C-true in K , then it contains a sentence which is not C-true in K ([I] T29-73) and which therefore is L-false in S . Hence, in this case, \mathfrak{R}_i itself is L-false in S , and therefore $\mathfrak{R}_i \vdash \mathfrak{S}_j$. Thus S is an L-true interpretation for K . — b. Let \mathfrak{S}_n be a sentence in K such that neither \mathfrak{S}_n nor $\text{neg}_C(\mathfrak{S}_n)$ is C-true in K , e.g. one of the propositional constants. Then both \mathfrak{S}_n and $\text{neg}_C(\mathfrak{S}_n)$ are L-false in S . Thus neg_C violates N2 with respect to \mathfrak{S}_n . — c. From T16-7.

Let us suppose that S' , in addition to the extensional connectives of NTT, contains non-extensional connectives, e.g. signs for logical necessity and for logical (strict) implication (compare [I] §§ 16 and 17). Let us designate the full sentence of the sign of necessity with \mathfrak{S}_i as component by 'nec(\mathfrak{S}_i)'. Then we might translate every sentence \mathfrak{S}_m in S into $\text{nec}(\mathfrak{S}'_m)$. This is essentially the same interpretation as that in the second example, because here, too, the C-true sentences in K are L-true in S , and the other sentences in K are L-false in S . $\text{imp}_C(\mathfrak{S}_i, \mathfrak{S}_j)$ is hereby translated into $\text{nec}(\text{imp}_L(\mathfrak{S}'_i, \mathfrak{S}'_j))$, which is L-equivalent to (and may be taken as definiens for) the sentence of logical (strict) implication with \mathfrak{S}'_i and \mathfrak{S}'_j as components. Hence the chief sign of implication_C in a sentence in K is here interpreted as the non-extensional connective of logical (strict) implication. This is possible because we have here no factual components.

As we have said, the two examples given are of a trivial nature. Now we shall construct examples of *non-trivial* non-normal interpretations. We shall not define the concept 'non-trivial interpretation'. The triviality meant here consists in the fact that too many sentences of K are interpreted in S as saying the same, i.e. are L-equivalent in S . Therefore it seems natural to take the following as a sufficient (though not necessary) condition for S to be a non-trivial interpretation for K : S is an interpretation for K , and for any \mathfrak{X}_i and \mathfrak{X}_j in K , if \mathfrak{X}_i and \mathfrak{X}_j are not C-equivalent in K ,

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they are not L-equivalent in S . The following examples fulfill this condition.

Third example: an L-true, non-normal interpretation of the first kind. We translate every sentence \mathcal{S}_i in S into $\text{dis}_L(\mathcal{S}'_i, \mathcal{S}_1)$. (For ' \mathcal{S}'_i ', see (H); for ' \mathcal{S}_1 ', (F).) Then the following holds: **a.** Every sentence of S is true. **b.** S is an L-true interpretation for K . **c.** neg_C in K violates N_1 in S . **d.** S is a non-normal interpretation of the first kind.

Proof. **a.** \mathcal{S}_1 is true in S' (F(c)). Hence, for every \mathcal{S}_i , $\text{dis}_L(\mathcal{S}'_i, \mathcal{S}_1)$ is true in S' (NTT). Hence, because of the translation, \mathcal{S}_i is true in S . — **b.** Let \mathcal{S}_k be a primitive sentence (i.e. a direct C-implicate of Λ , see [I] D28-10) in K . Then \mathcal{S}'_k is L-true by NTT in S' (T14-2a), and hence likewise $\text{dis}_L(\mathcal{S}'_k, \mathcal{S}_1)$ (T13-26b(1)). Therefore, because of the translation, \mathcal{S}_k is L-true in S . Let \mathcal{X}_m not be Λ , and $\mathcal{X}_m \xrightarrow{dC} \mathcal{S}_n$ in K . Then $\mathcal{X}'_m \xrightarrow{L} \mathcal{S}'_n$ by NTT in S' (T14-2b,c). Let \mathcal{X}''_m be that sentence or class into which \mathcal{X}_m is translated. [If \mathcal{X}_m is a class, \mathcal{X}''_m is the class constructed out of \mathcal{X}'_m by replacing every sentence \mathcal{S}'_m of \mathcal{X}'_m by $\text{dis}_L(\mathcal{S}'_m, \mathcal{S}_1)$.] Then $\mathcal{X}''_m \xrightarrow{L} \text{dis}_L(\mathcal{S}'_n, \mathcal{S}_1)$ in S' (T13-38b and 39). Therefore, because of the translation, $\mathcal{X}_m \xrightarrow{L} \mathcal{S}_n$ in S . Hence, S is an L-true interpretation for K . — **c** and **d.** As in the first example.

Fourth example: an L-true, non-normal interpretation of the second kind. A sentence \mathcal{S}_i in S is translated, if it is C-true in K , into \mathcal{S}'_i , and otherwise into $\text{con}_L(\mathcal{S}'_i, \mathcal{S}_2)$. Then the following holds: **a.** If \mathcal{S}_k is a primitive sentence in K , it is L-true in S . **b.** If \mathcal{X}_m is not Λ , and $\mathcal{X}_m \xrightarrow{dC} \mathcal{S}_n$ in K , then $\mathcal{X}_m \xrightarrow{L} \mathcal{S}_n$ in S . **c.** S is an L-true interpretation for K . **d.** neg_C in K violates N_2 in S . **e.** S is a non-normal interpretation of the second kind.

Proof. **a.** If \mathcal{S}_k is a primitive sentence in K , it is C-true in K and hence translated into \mathcal{S}'_k , which is L-true in S' (T14-2a). Therefore \mathcal{S}_k is L-true in S . — **b.** Let \mathcal{X}_m not be Λ , and $\mathcal{X}_m \xrightarrow{dC} \mathcal{S}_n$. Then $\mathcal{X}'_m \xrightarrow{L} \mathcal{S}'_n$ by NTT in S' (T14-2b,c). We may assume that neither \mathcal{S}_n nor any of the sentences of \mathcal{X}_m are C-true in K ; any other case can easily be reduced to a case of this kind. Then \mathcal{S}_n is translated into

$\text{con}(\mathcal{S}'_m, \mathcal{S}_2)$. Let \mathcal{I}''_m be that sentence or class into which \mathcal{I}_m is translated. [If \mathcal{I}_m is a class, \mathcal{I}''_m is the class constructed out of \mathcal{I}'_m by replacing every sentence \mathcal{S}'_m of \mathcal{I}'_m by $\text{con}_L(\mathcal{S}'_m, \mathcal{S}_2)$.] \mathcal{I}''_m L-implies the following \mathcal{I} : \mathcal{I}'_m ([I] P14-11, T13-26b(3), [I] P14-12), and hence \mathcal{S}'_m (see above, [I] P14-5), further \mathcal{S}_2 (T13-26b(4)), and hence $\text{con}_L(\mathcal{S}'_m, \mathcal{S}_2)$ ([I] P14-12, T13-27b(3)), into which \mathcal{S}_n is translated. Therefore, $\mathcal{I}_m \xrightarrow{L} \mathcal{S}_n$ in S . — c. From (a), (b), [I] D34-1. — d. Let \mathcal{S}_m be a sentence in K such that neither \mathcal{S}_m nor $\text{neg}_C(\mathcal{S}_m)$ is C-true in K , e.g. one of the propositional constants. Then \mathcal{S}_m is translated into $\text{con}_L(\mathcal{S}'_m, \mathcal{S}_2)$. This sentence is false in S' (NTT), since \mathcal{S}_2 is false in S' (F(d)). Therefore \mathcal{S}_m is false in S . $\text{neg}_C(\mathcal{S}_m)$ is translated into $\text{con}_L(\text{neg}_L(\mathcal{S}'_m), \mathcal{S}_2)$, which is likewise false in S' . Therefore, $\text{neg}_C(\mathcal{S}_m)$ is false in S . Thus neg_C violates N2 with respect to \mathcal{S}_m . — e. From T16-7.

§ 18. PC is not a Full Formalization of Propositional Logic

L-truth and L-implication in propositional logic, i.e. in a system containing NTT, are exhaustively represented in PC and thereby formalized. But not all logical properties of the connectives in NTT are represented in PC. If we could find a calculus K containing the connectives in such a way that every connective could only be interpreted normally (i.e. such that it would have a normal interpretation in any true interpretation of K and an L-normal interpretation in any L-true interpretation of K), then we should say that K is a full formalization of propositional logic. PC does not fulfill this requirement. The problem is whether any other calculus does.

The rules of NTT give an interpretation for the propositional connectives (more precisely, for the singular and binary extensional connectives) and thereby constitute propositional logic. The rules PC are constructed as a calculus for propositional logic; that is to say, they have the purpose of representing the logical properties of the connectives of propositional logic as far as these properties can be represented by a calculus, i.e. by the use of the formal syntactical method.

Let us examine the question whether PC fulfills this purpose. It seems to be the generally accepted opinion that it does. And, at the first glance, there seem to be good reasons for this opinion. In order to be more concrete, let us regard a calculus K and a semantical system S fulfilling the following conditions:

- A. K contains PC_1^D , and no other rules of deduction.
- B. K contains only the following sentences: 1. n propositional constants; 2. the molecular sentences constructed out of them with the help of the connectives of PC.
- C. The sentences of S are those of K .
- D. S contains NTT in such a way that the sign for a connection_C in K is simultaneously the sign for the corresponding connection_L of NTT in S .
- E. The truth-rules for the propositional constants in S are such that these sentences are mutually L-independent and hence factual (the further details of these truth-rules are irrelevant for the following discussion).

If a calculus is constructed as a formalization of logic within a certain region, then it is often regarded as its chief or even as its only purpose to present some or all L-true sentences of the region in question as C-true. In the case of K and S as specified, this task is fulfilled. Not only some but all L-true sentences of S are C-true in K (T_{14-5a}), and no others (T_{14-3b}). Thus C-truth in K is an exhaustive formalization of L-truth in S . Further, the formalization of logic, and analogously that of an empirical theory, in a certain region has a second task, which is sometimes overlooked; the calculus has to supply, in addition to suitable proofs, suitable derivations. In the case of a formalization of logic, some or all instances of L-implication have to be represented as instances of C-implication in the calculus. In our case, this second task also is fulfilled; C-implication in K has the same

extension as L-implication in S (T14-5c, T14-3a). In other words, the rules PC constitute an exhaustive formalization of logical deduction by NTT. Thus the rules PC, both in proofs and in derivations, yield all those and only those results for which they are made. What else could we require of them?

The statements just made concerning PC and its relation to NTT are correct. But the conclusion which seems to be generally, though tacitly, drawn from them — namely, that PC is a complete formal representation of propositional logic, i.e. of the logical properties of the propositional connectives in NTT — is wrong. This is shown by the possibility of non-normal interpretations. Thus, for instance, it belongs to the logical properties of disjunction in propositional logic that a sentence of disjunction with two false components is false (rule Dj4 in NTT, § 10). This property is not in any way represented in PC; this is shown by examples of true (and even L-true) interpretations of a calculus containing PC, in which the rule Dj4 is violated.

A full formalization of NTT would consist in a calculus K of such a kind that any connective of PC in K would have a normal interpretation in any true interpretation for K and an L-normal interpretation in any L-true interpretation for K . The problem is whether a full formalization of NTT in this sense is possible.

D. JUNCTIVES

If a full formalization of propositional logic is to be effected, new syntactical concepts must be used (§ 19). If rules of refutation are used and thereby 'C-false' is defined, the non-normal interpretations of PC of the first kind can be eliminated (§ 20). A more decisive change is made by the introduction of the junctives, i.e. of sentential classes in conjunctive and in disjunctive conception. Radical semantical concepts (§ 21) and L-concepts (§ 22) are defined for junctives. Further, junctives are applied in syntax; C-concepts are defined for them (§ 23). Their use in syntax makes possible a new kind of deductive rules, the disjunctive rules (§ 24). In this chapter, the general features of junctives and of calculi and semantical systems containing junctives are studied, leaving aside propositional logic and PC.

§ 19. Syntactical Concepts of a New Kind are Required

A calculus of the customary kind, consisting of primitive sentences and rules of inference, states conditions for C-implication (and C-truth) only. Therefore, it can formalize only those L-concepts which are definable on the basis of L-implication. 'L-true' belongs to these concepts, but 'L-exclusive' and 'L-disjunct' do not. Hence they cannot be formalized without the help of syntactical concepts of a new kind. The two concepts mentioned occur in the principles of contradiction and of the excluded middle. Therefore, these principles cannot be represented in PC. In a non-normal interpretation of the first kind, the first principle is violated; in one of the second kind, the second principle.

We found that PC does not completely fulfill its purpose; it is not a full formalization of propositional logic. This defect is by no means a particular feature of PC, however, but is based on general features of the customary method of

constructing calculi. This method consists in laying down rules for C-implication. Hence, on the basis of this method, a calculus can exhibit only those syntactical properties and relations of sentences which are definable by C-implication, above all C-truth. Therefore, a calculus of this customary kind, if constructed for the purpose of formalizing the logic of a certain region, can formalize only those logical properties and relations of sentences which are definable by L-implication, among them L-truth. We shall now examine some elementary logical relations with respect to the question whether they are definable by L-implication or not.

(a)		(b)		(c)	(d)	(e)
CONDITIONAL RELATION		It is not the case that		SEMANTICAL CONCEPTS		SYNTACTICAL CONCEPTS
If \mathcal{S}_i is	then \mathcal{S}_j is	\mathcal{S}_i is	and \mathcal{S}_j is	Radical Concepts	L-Concepts	C-Concepts
1. true	true	true	false	\mathcal{S}_i implies \mathcal{S}_j	L-implies	C-implies
2. true	false	true	true	\mathcal{S}_i is exclusive of \mathcal{S}_j	L-exclusive	C-exclusive
3. false	true	false	false	\mathcal{S}_i is disjunct with \mathcal{S}_j	L-disjunct	C-disjunct
4. false	false	false	true	\mathcal{S}_i is an implicate of \mathcal{S}_j	L-implicate	C-implicate

There are four elementary relations between two sentences which can be formulated by conditional statements with respect to their truth-values (see table, column (a)), or, more exactly, by statements excluding one of the four possible distributions of truth-values (column (b)). To the radical concepts (column (c), compare [I] D9-3, 6, and 5) there are corresponding L-concepts (column (d), compare [I] § 14; for 'L-disjunct', compare remarks in [I] § 14 and [I] D20-17). There could be corresponding syntactical C-concepts (column (e)). However, with respect to a calculus of the customary kind, we have only 'C-implies' and its inverse 'C-implicate', while 'C-exclusive' and 'C-disjunct' are not

definable by C-implication. Therefore the concepts 'L-exclusive' and 'L-disjunct' cannot be formalized in a calculus of the customary kind. We shall see that the circumstance that these two concepts are not represented is responsible for the possibility of non-normal interpretations of the first and second kind for the propositional connectives.

If we find that a certain calculus which has been constructed with regard to certain interpretations admits also of undesired interpretations, then we have to make the calculus stronger. In a situation of this kind, one usually thinks first of adding new primitive sentences or new rules of inference. But the defect here discussed cannot be removed in this way. It is well known that the rules of PC are already complete with respect to primitive sentences and rules of inference. Therefore, a full formalization of NTT, if it is at all possible, requires syntactical concepts of a new kind.

If a form K of PC is constructed with propositional variables as the only atomic sentences, then K is complete in the following sense with respect to direct C-implication, or, in other words, with respect to primitive sentences and rules of inference. If we construct a new calculus K' out of K by declaring any sentence \mathfrak{S}_i of K as an additional primitive sentence, then \mathfrak{S}_i is either already C-true in K or not. In the first case the addition is superfluous, because K' is coincident with K ([I], D31-9). In the second case K' becomes rather trivial because every sentence is C-true in K' , even those which are L-false in the normal interpretation. [In the customary terminology, K' is called contradictory or inconsistent in this case; but it is not C-inconsistent in our sense and still has true interpretations; see [I] D31-2 and remarks on [I] T31-31.] The same holds for the addition of a rule of inference.

If we take K and S as discussed in § 18 (fulfilling the conditions A to E), then any addition of a primitive sentence or a rule of inference would have the effect that there would be at least one \mathfrak{S}_i such that it was F-true in S and C-true in K , or \mathfrak{T}_i and \mathfrak{T}_i such that $\mathfrak{T}_i \vdash \mathfrak{T}_i$ in S and $\mathfrak{T}_i \vdash \mathfrak{T}_i$ in K , in contradiction to the intention of formalizing propositional logic.

In propositional logic, the sign of negation_L fulfills the following two principles (taken here in their semantical, as distinguished from their absolute, form) (T_{13-5b}(2)).

A. Principle of (Excluded) Contradiction. For any closed sentence \mathcal{S}_i , \mathcal{S}_i and $\text{neg}_L(\mathcal{S}_i)$ are L-exclusive. That is to say, the two sentences cannot both be true. (This is due to the rule N₁ for neg_L ; see T_{13-3b}(3).)

B. Principle of Excluded Middle. For any closed sentence \mathcal{S}_i , \mathcal{S}_i and $\text{neg}_L(\mathcal{S}_i)$ are L-disjunct. That is to say, the two sentences cannot both be false. (This is due to the rule N₂ for neg_L ; see T_{13-4b}(3).)

Do these two principles also hold for PC? In other words, are the two properties of neg_L which the principles state represented in PC? It seems to be the general belief that they are, because $\text{neg}_C(\text{con}_C(\mathcal{S}_i, \text{neg}_C(\mathcal{S}_i)))$ and $\text{dis}_C(\mathcal{S}_i, \text{neg}_C(\mathcal{S}_i))$ are C-true by PC. But the circumstance mentioned above, that 'C-exclusive' and 'C-disjunct' are not definable by 'C-implicate' and hence not definable with respect to PC, may evoke some doubt. And, in fact, the two principles do not hold for PC. Neither their validity nor their invalidity is assured by the rules of PC, because in some L-true interpretations, namely those with an L-normal interpretation of the connectives, the two principles hold, while in others they do not. In a non-normal interpretation of the first kind (T₁₆₋₆), \mathcal{S}_i and $\text{neg}_C(\mathcal{S}_i)$ are always both true; hence A is always violated, while B is always fulfilled. In a non-normal interpretation of the second kind (T₁₆₋₇), \mathcal{S}_i and $\text{neg}_C(\mathcal{S}_i)$ are sometimes — not always — both false, and always at least one of the two is false; hence B is sometimes violated, while A is always fulfilled. The C-truth of $\text{neg}_C(\text{con}_C(\mathcal{S}_i, \text{neg}_C(\mathcal{S}_i)))$ does not represent A; it would do so only if the L-normal interpretations of the connectives were assured by PC, which they are not; the same holds for $\text{dis}_C(\mathcal{S}_i, \text{neg}_C(\mathcal{S}_i))$ and B.

§ 20. C-Falsity

One new syntactical concept which might be added to those used in customary calculi is 'C-false'. It is defined on the basis of 'directly C-false', which is defined by rules of refutation. By adding a rule of this kind to PC, the non-normal interpretations of the first kind can be excluded.

Let us first discuss calculi in general and later apply the result to PC. The rules of a calculus of the customary kind determine only C-implication and thereby C-truth, but not C-falsity, which is not definable by C-implication. Therefore, if we look for new syntactical concepts, to be added to the customary ones, it seems natural to take C-falsity. We have seen previously that rules of a new kind are necessary for the introduction of this concept; we have called them rules of refutation ([I] § 26). The rules of refutation of a calculus K define 'directly C-false in K '. On the basis of this concept, we lay down the following definition ([I] D28-3):

+D20-1. \mathfrak{A}_i is C-false in $K =_{\text{Df}}$ there is a directly C-false \mathfrak{A}_j which is derivable from \mathfrak{A}_i .

The rules of deduction of the customary kind are not sufficient for formalizing falsity. Suppose we wish to make sure that the sentence \mathfrak{S}_1 in K is false in every true interpretation for K . On the customary basis, we cannot reach this aim even if K contains PC. We might perhaps try to do it by taking $\text{neg}_C(\mathfrak{S}_1)$ as an additional primitive sentence. This would indeed assure that $\text{neg}_C(\mathfrak{S}_1)$ was true in every true interpretation for K . But this does not help, because, as we have seen, the rules of PC do not exclude true interpretations in which $\text{neg}_C(\mathfrak{S}_1)$ and \mathfrak{S}_1 are both true.

By adding a suitable rule of refutation to PC we can exclude the possibility of non-normal interpretations of the first kind and hence assure the validity of the principle of contradiction. Let us consider a system S and a calculus K

as explained in § 18 (fulfilling the conditions A to E). According to our intention to formalize the logic in S , we wish to construct a calculus K' out of K by adding a rule of refutation in such a way that all those \mathfrak{T} which are L-false in S , and no others, are C-false in K' . The \mathfrak{T} which are L-false in S are those which are L-comprehensive in S ([I] T_{14-107b}), and hence those which are C-comprehensive in K ([I] D₃₀₋₆) because L-implication in S coincides with C-implication in K . But it would be unnecessary to declare all C-comprehensive \mathfrak{T} as directly C-false. It would suffice to take any one C-comprehensive sentence, say $\text{con}_C(\mathfrak{S}_1, \text{neg}_C(\mathfrak{S}_1))$, and lay down a rule in K' stating that this sentence is directly C-false; then all C-comprehensive \mathfrak{T} would be C-false in K' . But even this rule would be stronger than necessary. All we have to assure is that at least one sentence of K' becomes false. This cannot be done by a rule saying "at least one sentence of K' is directly C-false", because we must have a rule of refutation *defining* 'directly C-false' before we can make an existential statement concerning this concept. The simplest way is to lay down the following rule, R₁.

+R₂₀₋₁. V (and only V) is directly C-false in K' .

Then in every true interpretation for K' , V is false, and hence at least one sentence is false ([I] T₉₋₁). Thus, rule R₁ excludes non-normal interpretations of the first kind for K' .

A rule of refutation like R₁ is useful in connection with many calculi. T₁ shows that under certain conditions, which are also fulfilled by K and S as just discussed, the addition of R₁ has the effect that L-falsity in S is exhaustively formalized in K .

+T₂₀₋₁. Let the calculus K and the semantical system S contain the same sentences, and C-implication in K coincide with L-implication in S . Let K contain no rule of refutation, and K' be constructed out of K by adding the rule of refuta-

tion R₁. Let S contain at least one L-false \mathfrak{A}_i . Then C-falsity in K and L-falsity in S coincide.

Proof. Let the conditions be fulfilled. Then \mathfrak{A}_i is C-false in K' if and only if V is derivable from \mathfrak{A}_i in K' (D₁) and hence in K , hence if and only if $\mathfrak{A}_i \xrightarrow{C} V$ in K ([I] T_{29-54a}), hence if and only if $\mathfrak{A}_i \xrightarrow{L} V$ in S , hence if and only if \mathfrak{A}_i is L-comprehensive in S ([I] D₁₄₋₅), hence if and only if \mathfrak{A}_i is L-false in S ([I] T_{14-107b}).

On the basis of 'C-false in K' ' other concepts can be defined, among them 'C-exclusive in K' ' ([I] D₃₀₋₃). It can then be shown that, on the basis of rule R₁, for any \mathfrak{S}_i , \mathfrak{S}_i and $\text{neg}_C(\mathfrak{S}_i)$ are C-exclusive in K' . Thus the principle of contradiction holds for K' .

Later we shall introduce other syntactical concepts. With their help, 'C-false' will be definable on the basis of 'C-implicate' (D₂₃₋₆). Therefore, the concept 'directly C-false' will no longer be necessary. Rules of refutation, as e.g. rule R₁ above, will then be replaced by rules concerning 'direct C-implicate' (e.g. R₂₄₋₁) and thereby become analogous to the other rules of deduction.

§ 21. Junctives in Semantics

A sentential class is usually construed in the conjunctive way, i.e. as joint assertion of its sentences. Accordingly, \mathfrak{R}_i is regarded as true if and only if every sentence of \mathfrak{R}_i is true. However, a disjunctive conception is likewise possible. According to it, \mathfrak{R}_i is called true if and only if at least one sentence of \mathfrak{R}_i is true. The customary one-sided use of the conjunctive conception only is responsible for a lack of symmetry in the ordinary structure of syntactical and of semantical concepts. We begin here using both conceptions. If \mathfrak{R}_i is meant in the conjunctive way, it is called a conjunctive and designated by ' \mathfrak{R}_i^j '; if meant in the disjunctive way, it is called a disjunctive and designated by ' \mathfrak{R}_i^d '. Conjunctives, disjunctives, and sentences are together called junctives. Definitions and theorems concerning radical concepts ('true', etc.) with respect to junctives are stated.

In accordance with the customary use, we have construed sentential classes in such a way that asserting \mathfrak{R}_i means the same as asserting all sentences of \mathfrak{R}_i . Therefore we have called \mathfrak{R}_i true if and only if all sentences of \mathfrak{R}_i are true ([I] D9-1). Consequently, on the basis of NTT, a finite sentential class is L-equivalent with the conjunction_L of its sentences (e.g. $\{\mathfrak{S}_1, \mathfrak{S}_2\}$ is L-equivalent with $\text{con}_L(\mathfrak{S}_1, \mathfrak{S}_2)$, T13-14b). And to say that \mathfrak{S}_2 logically follows from \mathfrak{R}_1 (in our terminology, that $\mathfrak{R}_1 \xrightarrow{L} \mathfrak{S}_2$) means that, if every sentence of \mathfrak{R}_1 is true, \mathfrak{S}_2 is necessarily also true.

It would obviously also be possible, although not usual, to construe sentential classes in such a way that to assert \mathfrak{R}_i would mean the same as to assert that at least one of the sentences of \mathfrak{R}_i holds. If we adopted this way of using sentential classes, we should call \mathfrak{R}_i true if and only if at least one sentence of \mathfrak{R}_i was true. And a finite class would, in this case, be L-equivalent with the disjunction_L of its sentences.

The conjunctive conception of sentential classes seems

very convenient. We shall not replace it by the disjunctive conception but rather use both, distinguishing them with the help of two special signs. As previously, we shall use ' \mathfrak{R} ' with a subscript, e.g. ' \mathfrak{R}_2 ', as the designation of a class of sentences. \mathfrak{R}_i is a sentential class; it is determined, as every class is, with respect to the question of what elements (here sentences) belong to it; however, we shall regard it now as neutral with respect to the question how its assertion is to be construed. By ' \mathfrak{R}_i^* ' (read " \mathfrak{R}_i -con") we designate the class \mathfrak{R}_i as construed in the conjunctive way; by ' \mathfrak{R}_i^\vee ' (read " \mathfrak{R}_i -dis") we designate the class \mathfrak{R}_i as construed in the disjunctive way. \mathfrak{R}_i^* is called a conjunctive sentential class or, briefly, a **conjunctive**, \mathfrak{R}_i^\vee a disjunctive class or, briefly, a **disjunctive**. Conjunctives, disjunctives, and sentences (these we include for the sake of convenience in the formulation of definitions and theorems) are together called **junctives**. We have previously used ' \mathfrak{R} ' both for the neutral classes (e.g. " \mathfrak{R}_1 is a sub-class of \mathfrak{R}_2 ") and for the conjunctives (without this name) (e.g. " $\mathfrak{R}_2 \xrightarrow{L} \mathfrak{R}_1$ "); we shall use it in the remainder of this book for the neutral classes only. We have previously used ' \mathfrak{T} ' for sentences and sentential classes; we shall use it now for junctives in general. (Hence, "if \mathfrak{T}_i is false . . ." is to mean "if \mathfrak{S}_i , or \mathfrak{R}_i^* , or \mathfrak{R}_i^\vee is false . . .".)

It turns out that the customary tacit restriction of sentential classes to the conjunctive use is in fact the source of the lack of symmetry in the foundations of syntax and semantics, which we have often found in our previous discussions (e.g. in [I] pp. 38f, 72, 77, and 172; see, above, the remark concerning disjunction_c and conjunction_c at the beginning of § 16). By the use of both kinds of junctives, the foundations of semantics and likewise those of syntax will gain a perfect symmetry with respect to (L-, C-) truth and falsity, disjunction and conjunction, existential and universal sentences, etc.

The explanations above lead to the subsequent definitions for concepts applied to junctives: first their elements (D1 and 2, not often used), then truth (D3 and 4). For our purposes, it is not necessary to introduce the junctives themselves by explicit definitions. We simply assume that to every sentential class \mathfrak{R}_i two entities are correlated, which we designate by ' \mathfrak{R}_i^* ' and ' \mathfrak{R}_i^V '. And we shall define semantical concepts and later syntactical concepts applied to these entities by referring to the sentential class \mathfrak{R}_i .

An explicit definition of the junctives can easily be given if we construe them as ordered pairs. \mathfrak{R}_i^* might be regarded as the pair whose first member is \mathfrak{R}_i and whose second member is the connection of conjunction (hence as \mathfrak{R}_i^*); analogously \mathfrak{R}_i^V with disjunction. This procedure, however, presupposes that conjunction and disjunction are regarded as entities, say as relations between propositions; in other words, it presupposes the occurrence of (binary) connection variables in the metalanguage. But this difficulty can easily be avoided by taking any other two entities as second members of the pairs, e.g. the numbers 0 and 1, or the sentential classes V and Λ . In the latter case, $\mathfrak{R}_i^* = \mathfrak{R}_i; V$, and $\mathfrak{R}_i^V = \mathfrak{R}_i; \Lambda$. Here, the pairs are homogeneous.

D21-1. $x \in \mathfrak{R}_i^* =_{Df} x \in \mathfrak{R}_i$.

D21-2. $x \in \mathfrak{R}_i^V =_{Df} x \in \mathfrak{R}_i$.

+D21-3. \mathfrak{R}_i^* is *true* (in S) $=_{Df}$ every sentence of \mathfrak{R}_i is true.

+D21-4. \mathfrak{R}_i^V is *true* (in S) $=_{Df}$ at least one sentence of \mathfrak{R}_i is true.

D1 and 2 state that the elements of a conjunctive or disjunctive are the elements of the corresponding (neutral) sentential class; hence they are sentences. D3 and 4 take the place of [I] D9-1. The other definitions in [I] § 9 (for 'false', 'implicate', 'equivalent', etc.) are maintained in their previous form. Thus all radical semantical concepts can now be applied to junctives.

Junctives of higher levels could also be used, i.e. junctives containing other junctives as elements. We may even admit inhomogeneous

junctives, whose elements belong to different levels. Recursive definition for the level of a junctive:

D21-A1.

- a. The junctive \mathfrak{T}_i belongs to the first level $=_{Df}$ every element of \mathfrak{T}_i is a sentence.
- b. The junctive \mathfrak{T}_i belongs to the level $n + 1 =_{Df}$ at least one element of \mathfrak{T}_i belongs to the level n and none to a higher level.

The following definitions for 'true' (DA₃ and 4) are analogous to D₃ and 4. Thus the other radical concepts can also be applied analogously.

D21-A3. \mathfrak{T}_i^* is true $=_{Df}$ every element of \mathfrak{T}_i is true.

D21-A4. \mathfrak{T}_i^v is true $=_{Df}$ at least one element of \mathfrak{T}_i is true.

In the following discussions we shall restrict ourselves to junctives of the first level.

The following theorems are based on the definitions D₁ to 4. Those concerning conjunctives correspond exactly to certain theorems in the previous system ([I] § 9). Analogous theorems concerning disjunctives are added here; their proofs need not be given here, because they are analogous to the proofs for conjunctives, referring to the corresponding definitions and theorems for disjunctives.

+**T21-1.** \mathfrak{R}_i^* is false if and only if at least one sentence of \mathfrak{R}_i is false. ([I] T9-1.)

+**T21-2.** \mathfrak{R}_i^v is false if and only if every sentence of \mathfrak{R}_i is false.

T21-5. $\mathfrak{T}_i \rightarrow \mathfrak{R}_i^*$ if and only if \mathfrak{T}_i implies every sentence of \mathfrak{R}_i . ([I] T9-17.)

T21-6. $\mathfrak{R}_i^v \rightarrow \mathfrak{T}_i$ if and only if every sentence of \mathfrak{R}_i implies \mathfrak{T}_i .

The following theorems concern the null conjunctive Λ^* , the null disjunctive Λ^v , the universal conjunctive V^* , and the universal disjunctive V^v , with respect to a semantical system S .

T21-11. Λ^* is true. ([I] T9-32.)

T21-12. Λ^v is false. (From T2.)

T21-15. \mathfrak{A}_i is true if and only if $\Lambda^* \rightarrow \mathfrak{A}_i$. ([I] T9-35.)

T21-16. \mathfrak{A}_i is false if and only if $\mathfrak{A}_i \rightarrow \Lambda^v$.

T21-19.

a. V^* is true (in S) if and only if every sentence in S is true. ([I] T9-42a.)

b. V^* is false if and only if at least one sentence in S is false. ([I] T9-43a.)

T21-20.

a. V^v is true if and only if at least one sentence in S is true. (From D4.)

b. V^v is false if and only if every sentence in S is false. (From T2.)

T21-23. $\Lambda^* \rightarrow V^v$ if and only if at least one sentence in S is true. (From T15, T20a.)

+**T21-24.** $V^* \rightarrow \Lambda^v$ if and only if at least one sentence in S is false. (From T16, T19b.)

§ 22. Application of L-Concepts to Junctives

The two ways explained in [I] for introducing L-concepts are here adapted to junctives. 1. Eighteen postulates (P_1 to $_{15}$) are stated (corresponding to [I] P_{14-1} to $_{15}$), containing some of the L-concepts as primitives. A few theorems are based upon these postulates; among them: Λ^* is L-true (T22), Λ^v is L-false (T23). 2. The concept of L-range is applied to junctives (D1 and 2). On its basis, radical and L-concepts for junctives can be defined as previously (D11-5 to 8, and 12; [I] § 20). In this system, the postulates of the first system are provable.

In [I], the L-concepts were introduced in two different ways. Both of them can easily be adapted to junctives. The first way ([I] § 14) consisted in laying down fifteen postulates. Three of them ([I] P_{14-11} to $_{13}$) concern the

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relation between sentences and sentential classes; they must now be split up for conjunctives and disjunctives. The other postulates remain unchanged. Thus we come to the following system.

P22-1 to 10 (= [I] P₁₄₋₁ to 10).

+**P22-11.**

a. If $\mathfrak{S}_j \in \mathfrak{R}_i$, then $\mathfrak{R}_i^* \xrightarrow{L} \mathfrak{S}_j$.

b. If $\mathfrak{S}_i \in \mathfrak{R}_j$, then $\mathfrak{S}_i \xrightarrow{L} \mathfrak{R}_j^{\vee}$.

+**P22-12.**

a. If \mathfrak{T}_i L-implies every sentence of \mathfrak{R}_j , then $\mathfrak{T}_i \xrightarrow{L} \mathfrak{R}_j^*$.

b. If every sentence of \mathfrak{R}_i L-implies \mathfrak{T}_j , then $\mathfrak{R}_i^{\vee} \xrightarrow{L} \mathfrak{T}_j$.

P22-13.

a. If every sentence of \mathfrak{R}_i is L-true, \mathfrak{R}_i^* is L-true.

b. If every sentence of \mathfrak{R}_i is L-false, \mathfrak{R}_i^{\vee} is L-false.

P22-14 and 15 (= [I] P₁₄₋₁₄ and 15).

We give a few theorems based on these postulates. Those concerning conjunctives correspond exactly to theorems in the previous system. We add here theorems concerning disjunctives. They and their proofs are analogous to those concerning conjunctives.

T22-1. \mathfrak{S}_i and $\{\mathfrak{S}_i\}^*$ are L-equivalent. ([I] T₁₄₋₉.)

T22-2. \mathfrak{S}_i and $\{\mathfrak{S}_i\}^{\vee}$ are L-equivalent.

+**T22-3.** \mathfrak{S}_i , $\{\mathfrak{S}_i\}^*$, and $\{\mathfrak{S}_i\}^{\vee}$ are L-equivalent to one another. (From T₁ and 2.)

T22-6. If $\mathfrak{R}_j \subset \mathfrak{R}_i$, then $\mathfrak{R}_i^* \xrightarrow{L} \mathfrak{R}_j^*$. ([I] T₁₄₋₁₀.)

T22-7. If $\mathfrak{R}_i \subset \mathfrak{R}_j$, then $\mathfrak{R}_i^{\vee} \xrightarrow{L} \mathfrak{R}_j^{\vee}$.

T22-8. If a sentence of \mathfrak{R}_i is L-false, \mathfrak{R}_i^* is L-false. ([I] T₁₄₋₁₁.)

T22-9. If a sentence of \mathfrak{R}_i is L-true, \mathfrak{R}_i^{\vee} is L-true. (From P_{11b}, P6.)

T22-10. \mathfrak{R}_i^* is L-true if and only if every sentence of \mathfrak{R}_i is L-true. ([I] T14-20.)

T22-11. \mathfrak{R}_i^v is L-false if and only if every sentence of \mathfrak{R}_i is L-false. (From P13b; P7, P11b.)

+**T22-14.** $\mathfrak{X}_i \xrightarrow{L} \mathfrak{R}_i^*$ if and only if \mathfrak{X}_i L-implies every sentence of \mathfrak{R}_i . ([I] T14-22.)

+**T22-15.** $\mathfrak{R}_i^v \xrightarrow{L} \mathfrak{X}_j$ if and only if every sentence of \mathfrak{R}_i L-implies \mathfrak{X}_j . (From P12b; P11b, P5.)

T22-16.

a. $\mathfrak{R}_i^* \xrightarrow{L} \mathfrak{R}_i^*$. (From P11a, P12a.)

b. $\mathfrak{R}_i^v \xrightarrow{L} \mathfrak{R}_i^v$. (From P11b, P12b.)

T22-17. L-implication is reflexive; i.e., for every \mathfrak{X}_i , $\mathfrak{X}_i \xrightarrow{L} \mathfrak{X}_i$. (From P8, T16a,b.)

T22-18. If $\mathfrak{S}_i \xrightarrow{L} \mathfrak{S}_j$ (in S), then $\{\mathfrak{S}_i, \mathfrak{S}_k\}^* \xrightarrow{L} \{\mathfrak{S}_j, \mathfrak{S}_k\}^*$.

Proof. $\{\mathfrak{S}_i, \mathfrak{S}_k\}^*$ L-implies \mathfrak{S}_i (P11a) and hence \mathfrak{S}_j (P5), and likewise \mathfrak{S}_k , and hence $\{\mathfrak{S}_j, \mathfrak{S}_k\}^*$ (P12a).

T22-19. If $\mathfrak{S}_i \xrightarrow{L} \mathfrak{S}_j$ (in S), then $\{\mathfrak{S}_i, \mathfrak{S}_k\}^v \xrightarrow{L} \{\mathfrak{S}_j, \mathfrak{S}_k\}^v$.

Proof. $\{\mathfrak{S}_j, \mathfrak{S}_k\}^v$ is an L-implicate of \mathfrak{S}_k (P11b) and likewise of \mathfrak{S}_j , and hence of \mathfrak{S}_i (P5), and hence of $\{\mathfrak{S}_i, \mathfrak{S}_k\}^v$ (P12b).

T22-20. Every $\mathfrak{X}_i \xrightarrow{L} \Lambda^*$. ([I] T14-32.)

T22-21. $\Lambda^v \xrightarrow{L}$ every \mathfrak{X}_j . (From P12b.)

+**T22-22.** Λ^* is L-true. ([I] T14-33.)

+**T22-23.** Λ^v is L-false. (From P13b.)

+**T22-24.** \mathfrak{X}_j is L-true if and only if $\Lambda^* \xrightarrow{L} \mathfrak{X}_j$. ([I] T14-51a.)

+**T22-25.** \mathfrak{X}_i is L-false if and only if $\mathfrak{X}_i \xrightarrow{L} \Lambda^v$. (From T23, P7; P15.)

We found previously that, within the customary framework of concepts concerning sentences and sentential classes, 'L-true' can be defined on the basis of 'L-implication' ([I] D14-A) but 'L-false' cannot. T24 and 25 show that this

§ 22. APPLICATION OF L-CONCEPTS TO JUNCTIVES III
 asymmetry disappears if junctives are used. The same holds for the corresponding C-terms in syntax (see, below, D23-5 and 6).

T22-30. $V^* \xrightarrow{L}$ every \mathfrak{S}_i and every \mathfrak{R}_j^* . ([I] T14-42.)

T22-31. Every \mathfrak{S}_i and every $\mathfrak{R}_i^* \xrightarrow{L} V^*$. (From P11b, T7.)

T22-32. $V^* \xrightarrow{L}$ every non-empty \mathfrak{R}_j^* . (From P11a, P11b, P5.)

T22-33. Every non-empty $\mathfrak{R}_i^* \xrightarrow{L} V^*$. (From P11a, P11b, P5.)

The second way of the introduction of the L-concepts explained in [I] made use of the concept of *L-range* ([I] § 20; compare above § 11). This system can easily be modified so as to apply to junctives. Since we have previously based our system of propositional logic on the concept of L-range (§ 11), we shall use in our subsequent discussions of propositional logic containing junctives (§ 25) the system now to be explained. As its basis, we simply take the definition for the L-range of sentential classes (T11-1; [I] D20-1b) here applied to conjunctives (D1) and add an analogous definition for disjunctives (D2).

+**D22-1.** $Lr\mathfrak{R}_i^*$ (in S) =_{Dr} the product of the L-ranges of the sentences of \mathfrak{R}_i .

+**D22-2.** $Lr\mathfrak{R}_i^*$ (in S) =_{Dr} the sum of the L-ranges of the sentences of \mathfrak{R}_i .

The previous definitions for the L-concepts based on the concept of L-range remain unchanged ([I] § 20, some of them stated above as D11-5 to 9). Further, our present system is to contain the definition of 'true' (D11-12), based on 'L-range' in connection with 'rs', and the definitions of the other radical concepts based on 'true' ([I] D20-14 to 18). The resulting concept of truth for junctives is in accordance

with D₂₁₋₃ and 4 (T₄₀ and 41 below); therefore, the theorems in § 21 are valid in the present system.

T22-40. \mathfrak{R}_i^* is true if and only if every sentence of \mathfrak{R}_i is true. (From D₁, D₁₁₋₁₂.)

T22-41. \mathfrak{R}_i^v is true if and only if at least one sentence of \mathfrak{R}_i is true. (From D₂, D₁₁₋₁₂.)

In [I] § 20 we have seen that the system based on the concept of L-range contains among its theorems all the postulates of the earlier system concerning L-concepts ([I] P₁₄₋₁ to 15). Therefore, our present system contains as theorems those of the postulates stated above which correspond to [I] P₁₄₋₁ to 15; these are P₂₂₋₁ to 10, 11a, 12a, 13a, 14 and 15. But the same can easily be shown for the rest also, that is, P_{11b} (D₂, D₁₁₋₇), P_{12b} (D₂, D₁₁₋₇), and P_{13b} (D₂, D₁₁₋₆). Thus the present system contains all postulates P₂₂₋₁ to 15, and all theorems based upon them (T₁, etc., above).

T22-46. (Lemma for T_{23-11b}.) If S contains \mathfrak{T}_i and \mathfrak{T}_j , and \mathfrak{T}_j is not an L-implicate of \mathfrak{T}_i in S , then there is a class \mathfrak{M}_k of junctives in S which fulfills the following conditions:

- a. $\mathfrak{T}_i \in \mathfrak{M}_k$.
- b. If $\mathfrak{T}_m \in \mathfrak{M}_k$ and $\mathfrak{T}_m \xrightarrow{L} \mathfrak{T}_n$ in S , then $\mathfrak{T}_n \in \mathfrak{M}_k$.
- c. $\mathfrak{R}_m^* \in \mathfrak{M}_k$ if and only if every sentence of $\mathfrak{R}_m \in \mathfrak{M}_k$.
- d. $\mathfrak{R}_m^v \in \mathfrak{M}_k$ if and only if at least one sentence of $\mathfrak{R}_m \in \mathfrak{M}_k$.
- e. Not $\mathfrak{T}_j \in \mathfrak{M}_k$.

Proof. Let \mathfrak{T}_i not L-implicate \mathfrak{T}_j . Then not $R_i \subset R_j$ (D₁₁₋₇). Hence there is an s_i such that the following holds: 1. $s_i \in R_i$; 2. $s_i \notin R_j$. Let \mathfrak{M}_k be the class of all junctives \mathfrak{T}_k in S such that $s_i \in R_k$. Then \mathfrak{M}_k fulfills the conditions (a) to (e). (a) follows from (1). If $s_i \in R_m$ and $R_m \subset R_n$, then $s_i \in R_n$; hence (b). (c) follows from D₁; (d) from D₂; (e) from (2).

§ 23. Junctives in Syntax

If a conjunctive (or a sentential class) occurs as a C-implicans, we cannot eliminate it by referring to sentences only; likewise with a disjunctive as C-implicate. A rule of deduction stating a disjunctive as direct C-implicate is called a disjunctive rule. Definitions for 'C-implicate', 'C-true', 'C-false', 'C-equivalent' for junctives are given (D₄ to 7). These definitions have a form quite different from that of the former definitions for C-concepts ([I] § 28). But they fulfill the requirement of adequacy; that is to say, \mathfrak{T}_i is C-true in K if and only if \mathfrak{T}_i is true in every true interpretation for K , and analogously for the other C-concepts (T₁₅ to 18). And the new C-concepts are in accordance with the old ones as far as the latter go (T₄₁).

So far we have explained the use of junctives only in semantics. But they may also be used in syntax. Here their use leads to a new kind of rules of deduction. We shall see later that, by adding rules of this new kind to PC, it will be possible to exclude all non-normal interpretations and thus to reach our aim, a full formalization of propositional logic. In this and the next sections, however, we are not concerned with PC but with the use of junctives in calculi in general.

Against the use of junctives in the construction of a calculus, the objection might perhaps be raised that it involves a fundamental change in the method of dealing with calculi. Whereas in the usual method we seem to have to do merely with sentences and therefore can carry out all operations, namely proofs and derivations, entirely within the object language, after the introduction of junctives we shall have to operate in the metalanguage. In fact, however, there is no fundamental change of this kind. A closer examination shows that, in dealing with any calculus, even one of the usual kind, we must always make use of the metalanguage.

The metalanguage is first necessary for stating primitive sentences. Simply writing them down would not do, because in this way they

would be merely asserted but not specified as primitive. As to the rules of inference, it is even more obvious that the metalanguage is necessary for their formulation. Furthermore, if a derivation is to be given, it is necessary to indicate which of the sentences in the series are meant as premisses. Instead of saying explicitly: "The first ten sentences of this series are taken as premisses", we may, of course, use any other way of indicating the same on the basis of a suitable convention, e.g. by drawing a line under the tenth sentence. But then this line is a sign in the metalanguage, as are the assertion-sign and the signs 'Pp.', 'Dem.' in [Princ. Math.], the lines '———', '— — —', etc., in Frege's proofs, the signs '∴' and 'q.e.d.' sometimes used in mathematical proofs, and the like. And, further, it is necessary to speak about sentential classes, not only about sentences. This fact is often concealed by the customary way of formulation, which says "derivable from such and such premisses" instead of "derivable from the class of such and such premisses". The sentential classes in the usual method of calculi are what we now call conjunctives. The only new feature in the new method is the use of disjunctives in addition to conjunctives. Thus there is no fundamental change in method.

The radical semantical concepts are based on the concept of truth ([I] § 9). Thus, for the application of these concepts to junctives, it suffices to define 'true' for junctives (D21-3 and 4). For the application of the syntactical concepts, an analogous procedure is not possible. First, not even analogous theorems hold. In contradistinction to D21-4, \mathfrak{R}_i^v may be C-true, for instance by being declared directly C-true, without any sentence of \mathfrak{R}_i being C-true. Further, 'C-true' is not a sufficient basis for the definition of the other C-terms. We have seen that in the previous system of syntax ([I], §§28 to 32) many C-terms can be defined on the basis of 'C-implicate' but some cannot (e.g. 'C-false', 'C-disjunct', 'C-exclusive', [I] §§ 28 and 30). Now we shall see that, if junctives are used, 'C-implicate' is a sufficient basis for the other terms. Therefore, we have to introduce the junctives in syntax in connection with the concept of C-implication.

In the usual method of calculi, a sentential class, corre-

sponding to what we now call a conjunctive, is most often used as a C-implicans, i.e. as a class of premisses from which something is derived. In a case of this kind the reference to a sentential class is necessary; it cannot be replaced by a reference to sentences. Sometimes a sentential class occurs also as a C-implicate ('consequence-class', [Syntax] § 48). But in a case of this kind a reference to sentences would suffice. [Instead of saying: " \mathfrak{R}_2 (or, in the present terminology, \mathfrak{R}_2^*) is a C-implicate of \mathfrak{T}_1 ", we may say: "Every sentence of \mathfrak{R}_2 is a C-implicate of \mathfrak{T}_1 ", in analogy to T21-5.] Now we use disjunctives in addition to conjunctives. For them, the converse holds; reference to a disjunctive as C-implicans can be replaced by a reference to sentences, but reference to a disjunctive as C-implicate cannot. [Instead of saying: " \mathfrak{T}_2 is a C-implicate of \mathfrak{R}_1^* ", we may say: " \mathfrak{T}_2 is a C-implicate of every sentence of \mathfrak{R}_1 ", in analogy to T21-6.] A rule of deduction of the form " \mathfrak{R}_i^* is a direct C-implicate of \mathfrak{T}_i " cannot be expressed with the help of the usual syntactical concepts. We call a rule of deduction of this new kind, stating a disjunctive as a direct C-implicate of something, a **disjunctive rule** (of deduction).

If junctives are used, all rules of deduction of a calculus K can be stated in the same form, namely as parts of the definition for 'direct C-implicate in K '. We shall see that rules formulated in this way fulfill their purpose; that is, they have the effect that a certain intended result holds for every true interpretation of K . In order to show this, we must first define 'true interpretation' for systems containing junctives (D1a). The definition is similar to, but simpler than, [I] D33-2. The definition for 'L-true interpretation' (D1b) is analogous.

+D23-1a [b]. S is an [L-]true interpretation for $K =_{\text{Df}} S$ is an interpretation for K ([I] D33-1), and for every \mathfrak{T}_i and \mathfrak{T}_j , if $\mathfrak{T}_i \xrightarrow{\text{dc}} \mathfrak{T}_j$ in K , $\mathfrak{T}_i \xrightarrow{\text{ll}} \mathfrak{T}_j$ in S .

The *C-concepts for junctives*, in order to fulfill the requirement of adequacy ([I] § 28), must be defined on the basis of 'direct C-implication' in such a way that they apply in all those cases and only those in which the corresponding radical concepts apply in every true interpretation. [For instance, the definition of 'C-true' must be such that the following holds: \mathfrak{I}_i is C-true in K if and only if \mathfrak{I}_i is true in every true interpretation for K .] This condition of adequacy, however, uses semantical concepts and hence cannot itself be taken as a definition for the C-concepts. The task is to define these concepts in a purely syntactical way but such that the semantical condition just stated is fulfilled.

T23-1. Let S be a true interpretation for K . Let \mathfrak{M}_k be the class of those junctives in K which are true in S . Then \mathfrak{M}_k fulfills the following conditions:

- b. If $\mathfrak{I}_i \xrightarrow{\text{ac}} \mathfrak{I}_j$ in K , then, if $\mathfrak{I}_i \in \mathfrak{M}_k$, $\mathfrak{I}_j \in \mathfrak{M}_k$. (From D1b, [I] T9-10).
- c. $\mathfrak{R}_m^* \in \mathfrak{M}_k$ if and only if every sentence of $\mathfrak{R}_m \in \mathfrak{M}_k$. (From D21-3).
- d. $\mathfrak{R}_m^\vee \in \mathfrak{M}_k$ if and only if at least one sentence of $\mathfrak{R}_m \in \mathfrak{M}_k$. (From D21-4).

Our aim is to define C-implication so as to fulfill the condition of adequacy: $\mathfrak{I}_i \xrightarrow{c} \mathfrak{I}_j$ in K if and only if, in every true interpretation for K , $\mathfrak{I}_i \rightarrow \mathfrak{I}_j$, and hence, if \mathfrak{I}_i is true, \mathfrak{I}_j is true. Therefore we require in the following definition (D4) that for every \mathfrak{M}_k fulfilling the conditions (b), (c), and (d) in T1, if $\mathfrak{I}_i \in \mathfrak{M}_k$, $\mathfrak{I}_j \in \mathfrak{M}_k$. It will be seen later that these conditions are in fact sufficient to make the definition adequate (T15a).

+**D23-4.** \mathfrak{I}_j is a **C-implicate** of \mathfrak{I}_i ($\mathfrak{I}_i \xrightarrow{c} \mathfrak{I}_j$) (in K) =_{D1} \mathfrak{I}_j belongs to every class \mathfrak{M}_k of junctives which fulfills the following conditions:

- a. $\mathfrak{I}_i \in \mathfrak{M}_k$.

- b. If $\mathfrak{I}_m \in \mathfrak{M}_k$ and $\mathfrak{I}_m \xrightarrow{dC} \mathfrak{I}_n$, then $\mathfrak{I}_n \in \mathfrak{M}_k$.
 c. $\mathfrak{R}_m^* \in \mathfrak{M}_k$ if and only if every sentence of $\mathfrak{R}_m \in \mathfrak{M}_k$.
 d. $\mathfrak{R}_m^\vee \in \mathfrak{M}_k$ if and only if at least one sentence of $\mathfrak{R}_m \in \mathfrak{M}_k$.

T23-3. If $\mathfrak{I}_i \xrightarrow{dC} \mathfrak{I}_j$ in K , then $\mathfrak{I}_i \xrightarrow{C} \mathfrak{I}_j$ in K . (From D4a, b.)

+**T23-4.** C-implication is transitive, i.e. if $\mathfrak{I}_i \xrightarrow{C} \mathfrak{I}_j$ and $\mathfrak{I}_j \xrightarrow{C} \mathfrak{I}_l$, then $\mathfrak{I}_i \xrightarrow{C} \mathfrak{I}_l$.

Proof. Let $\mathfrak{I}_i \xrightarrow{C} \mathfrak{I}_j$ and $\mathfrak{I}_j \xrightarrow{C} \mathfrak{I}_l$. Let (a_i) be: $\mathfrak{I}_i \in \mathfrak{M}_k$; (a_j) : $\mathfrak{I}_j \in \mathfrak{M}_k$; (a_l) : $\mathfrak{I}_l \in \mathfrak{M}_k$; (b), (c), (d) as in D4. From the assumptions stated, we obtain by D4 the following. If \mathfrak{M}_k fulfills (a_i) , (b), (c), and (d), then also (a_j) ; if (a_j) , (b), (c), and (d), then also (a_l) . Hence, if \mathfrak{M}_k fulfills (a_i) , (b), (c), and (d), then also (a_l) . Thus $\mathfrak{I}_i \xrightarrow{C} \mathfrak{I}_l$ (D4).

+**T23-5.** C-implication is reflexive, i.e. $\mathfrak{I}_i \xrightarrow{C} \mathfrak{I}_i$. (From D4.)

Adequacy requires correspondence between C-concepts and radical semantical concepts. Therefore, the following definitions (D5, 6, and 7) are framed in analogy to T21-15 and 16, and [I] T9-2ob. We shall see later that the concepts thus defined are indeed adequate (T16, 17, and 18).

+**D23-5.** \mathfrak{I}_i is **C-true** (in K) =_{DI} $\Lambda^* \xrightarrow{C} \mathfrak{I}_i$.

+**D23-6.** \mathfrak{I}_i is **C-false** (in K) =_{DI} $\mathfrak{I}_i \xrightarrow{C} \Lambda^*$.

+**D23-7.** \mathfrak{I}_i is **C-equivalent** to \mathfrak{I}_j (in K) =_{DI} $\mathfrak{I}_i \xrightarrow{C} \mathfrak{I}_j$ and $\mathfrak{I}_j \xrightarrow{C} \mathfrak{I}_i$.

Thus here, not only 'C-true' but also 'C-false' is definable on the basis of 'C-implication'.

+**T23-11a [b].** If S is an [L-]true interpretation for K and $\mathfrak{I}_i \xrightarrow{C} \mathfrak{I}_j$ in K , then $\mathfrak{I}_i \xrightarrow{[L]} \mathfrak{I}_j$ in S .

Proof for (a). Let the conditions be fulfilled, and \mathfrak{M}_k be the class of the junctives in K which are true in S . Then \mathfrak{M}_k fulfills the conditions D4b, c, d (T1). — 1. Let \mathfrak{I}_i be false in S . Then $\mathfrak{I}_i \rightarrow \mathfrak{I}_j$ ([I] T9-12). — 2. Let \mathfrak{I}_i be true in S . Then $\mathfrak{I}_i \in \mathfrak{M}_k$. Hence, \mathfrak{M}_k ful-

fills also D4a. Therefore, $\mathfrak{T}_j \in \mathfrak{M}_k$ (D4); \mathfrak{T}_j is true; $\mathfrak{T}_i \rightarrow \mathfrak{T}_j$ ([I] T9-13). — *Proof for (b).* Let the conditions be fulfilled. Let S' be the sub-system of S which contains the junctives of K only. Then S' is also an L-true interpretation for K . For the sake of an indirect proof, let us suppose that \mathfrak{T}_j is not an L-implicate of \mathfrak{T}_i in S' . Then there would be an \mathfrak{M}_k which fulfills the conditions (a) to (e) in T22-46 with respect to S' , \mathfrak{T}_i , and \mathfrak{T}_j . Then \mathfrak{M}_k would fulfill the conditions (a), (c), and (d) in D4 with respect to \mathfrak{T}_i and K . But it would also fulfill D4b; for, if $\mathfrak{T}_m \in \mathfrak{M}_k$, and $\mathfrak{T}_m \xrightarrow{dC} \mathfrak{T}_n$ in K , then $\mathfrak{T}_m \xrightarrow{I} \mathfrak{T}_n$ in S' (D1b), hence $\mathfrak{T}_n \in \mathfrak{M}_k$ (T22-46b). Therefore, since $\mathfrak{T}_i \xrightarrow{C} \mathfrak{T}_j$, $\mathfrak{T}_j \in \mathfrak{M}_k$ (D4); but also \mathfrak{T}_j not $\in \mathfrak{M}_k$ (T22-46e). Thus our supposition is impossible. $\mathfrak{T}_i \xrightarrow{I} \mathfrak{T}_j$ in S' and hence in S .

T23-13 (lemma). Let \mathfrak{M}_k be a class of junctives in S which fulfills the conditions (c) and (d) in D4.

- a. If every sentence in \mathfrak{M}_k is true (in S), then every junctive in \mathfrak{M}_k is true.
- b. If every sentence in S which does not belong to \mathfrak{M}_k is false (in S), then every junctive in S which does not belong to \mathfrak{M}_k is false.

Proof. a. Let the conditions be fulfilled. Let $\mathfrak{R}_i^* \in \mathfrak{M}_k$. Then every sentence of $\mathfrak{R}_i^* \in \mathfrak{M}_k$ (c) and hence is true. Therefore, \mathfrak{R}_i^* is true (D21-3). Let $\mathfrak{R}_i^y \in \mathfrak{M}_k$. Then at least one sentence of $\mathfrak{R}_i^y \in \mathfrak{M}_k$ (d) and hence is true. Therefore, \mathfrak{R}_i^y is true (D21-4). — b. Let the conditions be fulfilled. Let \mathfrak{R}_j^* not $\in \mathfrak{M}_k$. Then there is a sentence \mathfrak{S}_j of \mathfrak{R}_j^* such that \mathfrak{S}_j not $\in \mathfrak{M}_k$ (c), and hence \mathfrak{S}_j is false. Therefore, \mathfrak{R}_j^* is false (T21-1). Let \mathfrak{R}_j^y not $\in \mathfrak{M}_k$. Then every sentence of \mathfrak{R}_j^y not $\in \mathfrak{M}_k$ (d) and hence is false. Therefore, \mathfrak{R}_j^y is false (T21-2).

T23-14 (lemma). If \mathfrak{T}_i and \mathfrak{T}_j are junctives in K and not $\mathfrak{T}_i \xrightarrow{C} \mathfrak{T}_j$ in K , then there is a system S such that the following holds:

- a. S is a true interpretation for K ,
- b. \mathfrak{T}_i is true in S ,
- c. \mathfrak{T}_j is false in S .

Proof. Let K , \mathfrak{T}_i , and \mathfrak{T}_j fulfill the conditions. Then, according to D4, there is a class \mathfrak{M}_k such that 1. \mathfrak{M}_k and \mathfrak{T}_i fulfill the conditions D4a, b, c, d; 2. not $\mathfrak{T}_j \in \mathfrak{M}_k$. Now we construct S in the following

way: 3. S contains the same sentences as K ; 4. every sentence (not conjunctive or disjunctive) of \mathcal{M}_k is true in S ; 5. every other sentence is false in S . [Since, in constructing a semantical system, we can freely choose the truth-conditions for the sentences, we can obtain the results (4) and (5) simply by laying down, for instance, the rules that any sentence of \mathcal{M}_k designates the L-true proposition, i.e. that it is true if and only if A or not A , and that any other sentence is true if and only if A and not A .] Then the following holds: 6. Every junctive in \mathcal{M}_k is true in S ((1), (4), T13a). 7. Every other junctive in S is false ((1), (5), T13b). 8. Let \mathfrak{T}_m and \mathfrak{T}_n be any junctives in K and hence in S such that $\mathfrak{T}_m \xrightarrow{dC} \mathfrak{T}_n$. We distinguish two cases, A and B. A. Let \mathfrak{T}_m not $\in \mathcal{M}_k$. Then \mathfrak{T}_m is false in S (7), and hence $\mathfrak{T}_m \rightarrow \mathfrak{T}_n$ ([I] D9-3). B. Let $\mathfrak{T}_m \in \mathcal{M}_k$. Then $\mathfrak{T}_n \in \mathcal{M}_k$ ((1), D4b) and hence is true (6). Therefore, $\mathfrak{T}_m \rightarrow \mathfrak{T}_n$. 9. (a) from D1, (3), (8). 10. $\mathfrak{T}_i \in \mathcal{M}_k$ ((1), D4a), and hence is true (6). This is (b). 11. (c) from (2), (7).

+T23-15. $\mathfrak{T}_i \xrightarrow{C} \mathfrak{T}_j$ in K if and only if, for every true interpretation S for K , $\mathfrak{T}_i \rightarrow \mathfrak{T}_j$ in S .

Proof. 1. From T11a. — 2. If $\mathfrak{T}_i \rightarrow \mathfrak{T}_j$ in every true interpretation for K , then there is no true interpretation in which \mathfrak{T}_i is true and \mathfrak{T}_j false ([I] T9-18). Therefore, $\mathfrak{T}_i \xrightarrow{C} \mathfrak{T}_j$ in K (T14).

+T23-16. \mathfrak{T}_i is C-true in K if and only if \mathfrak{T}_i is true in every true interpretation for K . (From D5, T15, T21-15.)

+T23-17. \mathfrak{T}_i is C-false in K if and only if \mathfrak{T}_i is false in every true interpretation for K . (From D6, T15, T21-16.)

+T23-18. \mathfrak{T}_i is C-equivalent to \mathfrak{T}_j in K if and only if \mathfrak{T}_i is equivalent to \mathfrak{T}_j in every true interpretation for K . (From D7, T15, [I] T9-20b.)

+T23-19. If S is an L-true interpretation for K , then the following holds:

- a. If \mathfrak{T}_i is C-true in K , it is L-true in S . (From D5, T11b, T22-24.)
- b. If \mathfrak{T}_i is C-false in K , it is L-false in S . (From D6, T11b, T22-25.)
- c. If \mathfrak{T}_i and \mathfrak{T}_j are C-equivalent in K , then they are L-equivalent in S . (From D7, T11b, P22-9.)

+**T23-21.** If $\mathfrak{X}_i \supset_c \mathfrak{X}_j$ and \mathfrak{X}_i is C-true, then \mathfrak{X}_j is also C-true. (From D5, T4.)

+**T23-22.** If $\mathfrak{X}_i \supset_c \mathfrak{X}_j$ and \mathfrak{X}_j is C-false, then \mathfrak{X}_i is also C-false. (From D6, T4.)

Once the correspondence between C-concepts and radical concepts is proved (T15 to 18), further theorems concerning C-concepts in analogy to those concerning radical concepts can easily be proved (e.g. T23 to 26).

+**T23-23.** $\mathfrak{X}_i \supset_c \mathfrak{R}_j^*$ (in K) if and only if $\mathfrak{X}_i \supset_c$ every sentence of \mathfrak{R}_j . (From T15, T21-5.)

+**T23-24.** $\mathfrak{R}_i^* \supset_c \mathfrak{X}_j$ (in K) if and only if every sentence of $\mathfrak{R}_i \supset_c \mathfrak{X}_j$. (From T15, T21-6.)

T23-25. Every $\mathfrak{X}_i \supset_c \Lambda^*$. (From T23.)

T23-26. $\Lambda^* \supset_c$ every \mathfrak{X}_j . (From T24.)

T23-27. If \mathfrak{X}_j is C-true, every $\mathfrak{X}_i \supset_c \mathfrak{X}_j$. (From D5, T25, T4.)

T23-28. If \mathfrak{X}_i is C-false, $\mathfrak{X}_i \supset_c$ every \mathfrak{X}_j . (From D6, T26, T4.)

+**T23-30.** Λ^* is C-true. (From D5, T5.)

+**T23-31.** Λ^* is C-false. (From D6, T5.)

+**T23-34.** \mathfrak{X}_j is C-true (in K) if and only if every $\mathfrak{X}_i \supset_c \mathfrak{X}_j$. (From D5, T27.)

+**T23-35.** \mathfrak{X}_i is C-false (in K) if and only if $\mathfrak{X}_i \supset_c$ every \mathfrak{X}_j . (From D6, T28.)

The definition for 'C-implication' given here (D4) for calculi containing junctives has a form quite different from the definition of the same term for calculi of the customary kind ([I] D28-4). Nevertheless, the new concept is in accordance with the old one, as far as the latter goes. This is shown by T41, based on the lemma T40.

T23-40. (Lemma.) Let the calculi K_m and K_n and the class \mathfrak{M}_k fulfill the following conditions A to E.

- A. K_m is a calculus of the customary kind (as described in [I] § 28), i.e. the rules of K_m refer only to sentences and sentential classes but not to junctives.
- B. K_n is a calculus with junctives, containing all sentences of K_m .
- C. If $\mathfrak{I}_i \xrightarrow{\text{dC}} \mathfrak{S}_j$ in K_m , then $\mathfrak{I}^* \xrightarrow{\text{dC}} \mathfrak{S}_j$ in K_n . (If \mathfrak{I}_i is \mathfrak{R}_i or \mathfrak{S}_i , ' \mathfrak{I}_i^* ' means \mathfrak{R}_i^* or \mathfrak{S}_i respectively.)
- D. If \mathfrak{I}_i is directly C-false in K_m , $\mathfrak{I}_i^* \xrightarrow{\text{dC}} \Lambda^*$ in K_n .
- E. \mathfrak{M}_k is any class of junctives in K_n fulfilling the conditions (b), (c), and (d) in D4.

Then the following holds:

- a. If \mathfrak{I}_j is derivable from \mathfrak{I}_i in K_m (in the sense of [I] D28-2) and $\mathfrak{I}^* \in \mathfrak{M}_k$, then $\mathfrak{I}_j^* \in \mathfrak{M}_k$.
- b. Λ^* not $\in \mathfrak{M}_k$.
- c. If \mathfrak{I}_i is C-false in K_m , \mathfrak{I}_i^* not $\in \mathfrak{M}_k$.
- d. If $\mathfrak{I}_i \xrightarrow{\text{C}} \mathfrak{I}_j$ in K_m , and $\mathfrak{I}_i^* \in \mathfrak{M}_k$, then $\mathfrak{I}_j^* \in \mathfrak{M}_k$.
- e. $\Lambda^* \in \mathfrak{M}_k$.
- f. If \mathfrak{I}_j is C-true in K_m , then $\mathfrak{I}_j^* \in \mathfrak{M}_k$.

Proof. a. Let \mathfrak{S}_i be derivable from \mathfrak{R}_i in K_m . Then there is a sequence of sentences \mathfrak{R}_l ([I] D28-1) which fulfills the following conditions (F) and (G). F. For every sentence \mathfrak{S}_n in \mathfrak{R}_l not belonging to \mathfrak{R}_i , there is a sub-class \mathfrak{R}_p of the class \mathfrak{R}_n of the sentences preceding \mathfrak{S}_n in \mathfrak{R}_l such that $\mathfrak{R}_p \xrightarrow{\text{dC}} \mathfrak{S}_n$ in K_n . G. \mathfrak{S}_j is the last sentence in \mathfrak{R}_l . Let $\mathfrak{R}_i \in \mathfrak{M}_k$. Then every sentence of \mathfrak{R}_i and hence the conjunctive of every sub-class of \mathfrak{R}_i belong to \mathfrak{M}_k (E(c)). If \mathfrak{S}_m is the first sentence of \mathfrak{R}_l which does not belong to \mathfrak{R}_i , then \mathfrak{S}_m is a direct C-implicate in K_m of a sub-class of \mathfrak{R}_i (F) and hence $\mathfrak{S}_m \in \mathfrak{M}_k$ (C, E(b)). Let \mathfrak{S}_n be any sentence in \mathfrak{R}_l but not in \mathfrak{R}_i and let \mathfrak{R}_p and \mathfrak{R}_n be as above (F). Then, if every sentence of \mathfrak{R}_n belongs to \mathfrak{M}_k , $\mathfrak{R}_p^* \in \mathfrak{M}_k$ (E(c)) and $\mathfrak{S}_n \in \mathfrak{M}_k$ (E(b)). Therefore, by induction, every sentence of \mathfrak{R}_l belongs to \mathfrak{M}_k , hence also \mathfrak{S}_j (G). Let \mathfrak{R}_j be derivable from \mathfrak{R}_i in K_m . Then, for every sentence \mathfrak{S}_i of \mathfrak{R}_j , \mathfrak{S}_i is derivable from \mathfrak{R}_i in K_m ([I] D28-2b) and, according to the result just found, $\mathfrak{R}_i^* \xrightarrow{\text{C}} \mathfrak{S}_i$ in K_n , and hence $\mathfrak{R}_i^* \xrightarrow{\text{C}} \mathfrak{R}_j^*$ in K_n (T23). The results hold likewise for \mathfrak{S}_i instead of \mathfrak{R}_i ([I], D28-2c). — b. From (E), D4d, since Λ has no ele-

ment. — c. Let \mathfrak{X}_i be C-false in K_m . Then ([I] D28-3) there is a directly C-false \mathfrak{X}_p which is derivable from \mathfrak{X}_i in K_m . If \mathfrak{X}_i^* were an element of \mathfrak{M}_k , then \mathfrak{X}_p would also be one (a). Further, $\mathfrak{X}_p \xrightarrow{dC} \Lambda^v$ in K_n (D); hence (E(b)) Λ^v would be an element of \mathfrak{M}_k , which is impossible (b). Therefore, \mathfrak{X}_i^* not $\in \mathfrak{M}_k$. — d. Let $\mathfrak{X}_i \xrightarrow{C} \mathfrak{X}_j$ in K_m . Then ([I] D28-4) either \mathfrak{X}_j is derivable from \mathfrak{X}_i or \mathfrak{X}_i is C-false in K_m . Let $\mathfrak{X}_i^* \in \mathfrak{M}_k$. Then \mathfrak{X}_i cannot be C-false in K_m (c). Hence \mathfrak{X}_j is derivable from \mathfrak{X}_i ; hence $\mathfrak{X}_j^* \in \mathfrak{M}_k$ (a). — e. From (E), D4c. — f. Let \mathfrak{X}_i be C-true in K_m . Then $\Lambda \xrightarrow{C} \mathfrak{X}_i$ in K_m ([I] D28-5). Therefore, since $\Lambda^* \in \mathfrak{M}_k$ (e), $\mathfrak{X}_i^* \in \mathfrak{M}_k$ (d).

+T23-41. Let K_m and K_n fulfill the conditions (A), (B), (C), and (D) in T40. Then the following holds:

- a. If $\mathfrak{X}_i \xrightarrow{C} \mathfrak{X}_j$ in K_m , then $\mathfrak{X}_i^* \xrightarrow{C} \mathfrak{X}_j^*$ in K_n .
- b. If \mathfrak{X}_j is C-true in K_m , \mathfrak{X}_j^* is C-true in K_n .
- c. If \mathfrak{X}_i is C-false in K_m , \mathfrak{X}_i^* is C-false in K_n .

Proof. a. From T40d, D4. — b. Let \mathfrak{X}_j be C-true in K_m . Then $\Lambda \xrightarrow{C} \mathfrak{X}_j$ in K_m ([I] D28-5). Hence $\Lambda^* \xrightarrow{C} \mathfrak{X}_j^*$ in K_n (a); hence \mathfrak{X}_j^* is C-true in K_n (D5). — c. Let \mathfrak{X}_i be C-false in K_m . Then Λ^v belongs to every class \mathfrak{M}_k which contains \mathfrak{X}_i^* and fulfills (b), (c), and (d) in D4, because there is no such \mathfrak{M}_k (T40c). Hence $\mathfrak{X}_i^* \xrightarrow{C} \Lambda^v$ (D4), and \mathfrak{X}_i^* is C-false (D6).

§ 24. Rules of Deduction for Junctives

In a calculus of the ordinary kind (without junctives), the statement of the primitive sentences as well as that of the rules of inference can be formulated as parts of the definition of 'direct C-implicate', while the formulation of the rules of refutation requires a new basic concept 'directly C-false' ([I] § 28). In a calculus with junctives, all rules of deduction can be formulated as conditions for direct C-implication. To these rules belong those just mentioned and, furthermore, several kinds of disjunctive rules. Among them, the rule " $V \xrightarrow{dC} \Lambda^v$ " (Rr) is of special interest. It has the effect that, in every true interpretation, at least one sentence is false (Tr8e).

Let us consider how rules of deduction of different kinds can be formulated in such a way that they state conditions

for direct C-implication. The purpose of laying down a rule of deduction in a calculus K is to make sure that certain conditions with respect to the truth and falsity of the sentences in K are fulfilled in every true interpretation for K . It will be shown that the rules described serve this purpose.

1. Suppose we wish a certain sentence, say \mathfrak{S}_1 , to be a *primitive sentence* in K . Our aim herein is to make sure that \mathfrak{S}_1 becomes true in every true interpretation for K . If we lay down the rule: " $\Lambda^* \xrightarrow{dC} \mathfrak{S}_1$ ", this aim is reached (T1a). A similar rule is used if every sentence of a certain kind is intended to become true (T1b); there may be an infinite number of such sentences.

T24-1.

- a. If $\Lambda^* \xrightarrow{dC} \mathfrak{S}_i$ in K , then in every true interpretation for K , \mathfrak{S}_i is true.
- b. If $\Lambda^* \xrightarrow{dC} \mathfrak{R}_i^*$ in K , then in every true interpretation for K , \mathfrak{R}_i^* is true and hence every sentence of \mathfrak{R}_i is true.

Proof. Let S be a true interpretation for K . a. Let $\Lambda^* \xrightarrow{dC} \mathfrak{S}_i$ in K . Then $\Lambda^* \rightarrow \mathfrak{S}_i$ in S (D23-1), and hence \mathfrak{S}_i is true in S (T21-15). — b. Let $\Lambda^* \xrightarrow{dC} \mathfrak{R}_i^*$ in K . Then \mathfrak{R}_i^* is true in S (as in (a)), and hence every sentence of \mathfrak{R}_i is true (D21-3).

2. A *rule of inference of the ordinary kind* is formulated here in the ordinary way, except that a conjunctive is taken instead of the class of premisses, e.g. " $\mathfrak{R}_1^* \xrightarrow{dC} \mathfrak{S}_2$ ". Here it is easily seen that the purpose is fulfilled (T3a). Analogously for a class of several conclusions (T3b).

T24-3a [b]. Let $\mathfrak{R}_i^* \xrightarrow{dC} \mathfrak{S}_j [\mathfrak{R}_j^*]$. Let S be a true interpretation for K , and all sentences of \mathfrak{R}_i be true in S . Then \mathfrak{S}_j [every sentence of \mathfrak{R}_j] is true in S .

Proof for a [b]. Let the conditions be fulfilled. Then $\mathfrak{R}_i^* \rightarrow \mathfrak{S}_j$ [\mathfrak{R}_j^*] in S (D23-1). \mathfrak{R}_i^* is true in S (D21-3). Therefore \mathfrak{S}_j [\mathfrak{R}_j^*] is true ([I] T9-10) [and hence every sentence of \mathfrak{R}_j (D21-3)].

3. Disjunctive rules

a. Suppose we want to ensure that, in every true interpretation for K , at least one of a given class of sentences, say \mathfrak{R}_1 , is true. Then we lay down the rule: " $\Lambda^* \xrightarrow{\text{dC}} \mathfrak{R}_1^v$ " (T5). Here, we may call \mathfrak{R}_1^v a *primitive disjunctive*, in analogy to the term 'primitive sentence'. If those sentences are finite in number, and each of them is known, say \mathfrak{S}_1 , \mathfrak{S}_2 , and \mathfrak{S}_3 , then the rule is: " $\Lambda^* \xrightarrow{\text{dC}} \{\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3\}^v$ ".

+T24-5. Let $\Lambda^* \xrightarrow{\text{dC}} \mathfrak{R}_j^v$ in K , and S be a true interpretation for K . Then at least one sentence of \mathfrak{R}_j is true in S . (From D23-1, T21-15, D21-4.)

Example of a primitive disjunctive. Hempel ("A Purely Topological Form of Non-Aristotelian Logic", *Journ. Symb. Logic*, vol. 2, 1937, p. 97; shorter representation in *Erkenntnis*, vol. 6, 1937, p. 436) constructs a language T of the following kind. T contains neither variables nor connectives. There are certain classes of three sentences each — we call the class of these classes \mathfrak{M}_1 — such that there is a true sentence in each of these classes. Hempel constructs a calculus — we call it K — for the language T . He remarks correctly that a rule of the ordinary kind determining the concept of direct consequence or consequence (i.e. direct C-implication or C-implication) does not suffice to represent the fact that there is a true sentence in every class of \mathfrak{M}_1 . Therefore, he lays down a rule (6.6, p. 106) concerning not 'consequence' but the concept 'closed system' (in our terminology, 'C-complete, C-perfect sentential class', [1] D30-5 and 7). Although this rule is stronger than a rule of the ordinary kind, it does not suffice to ensure that, in every true interpretation for K , there is a true sentence in every class of \mathfrak{M}_1 . (The reason for this is that there is not necessarily a state-description in S for every L -state; see [1] § 18 at the end.) This can, however, be done by the following disjunctive rule: "For every class \mathfrak{R}_i of \mathfrak{M}_1 , \mathfrak{R}_i^v is a primitive disjunctive in K (i.e. $\Lambda^* \xrightarrow{\text{dC}} \mathfrak{R}_i^v$)".

b. *Disjunctive rule of inference.* Suppose we want to ensure that, if \mathfrak{T}_1 is true in any true interpretation S for K , at least one of the sentences of a certain (finite or infinite) class \mathfrak{R}_2 is also true. We do this by the rule: " $\mathfrak{T}_1 \xrightarrow{\text{dC}} \mathfrak{R}_2^v$ ".

T24-7. Let $\mathfrak{T}_i \xrightarrow{\text{dC}} \mathfrak{R}_j^v$ in K . Let S be a true interpreta-

tion for K , and \mathfrak{X}_i be true in S . Then at least one sentence of \mathfrak{R}_j is true in S . (From D23-1, [I] T9-10, D21-4.)

c. A *rule of refutation* has the purpose of ensuring that one or several sentences are false in every true interpretation. If we want \mathfrak{S}_1 to become false, we lay down the rule of refutation: " $\mathfrak{S}_1 \xrightarrow{\text{ad}} \Lambda^\vee$ ". If every sentence of a given class \mathfrak{R}_1 is to become false, we state the rule: " $\mathfrak{R}_1^\vee \xrightarrow{\text{ad}} \Lambda^\vee$ ". If at least one of the sentences of \mathfrak{R}_1 is to become false, we state: " $\mathfrak{R}_1^\circ \xrightarrow{\text{ad}} \Lambda^\vee$ ".

+T24-9. Let $\mathfrak{X}_i \xrightarrow{\text{ad}} \Lambda^\vee$ in K . Let S be a true interpretation for K . Then the following holds:

- a. \mathfrak{X}_i is false in S . (From D23-1, T21-16.)
- b. If \mathfrak{X}_i is \mathfrak{R}_i^\vee , then every sentence of \mathfrak{R}_i is false in S . (From (a), T21-2.)
- c. If \mathfrak{X}_i is \mathfrak{R}_i° , then at least one sentence of \mathfrak{R}_i is false in S . (From (a), T21-1.)

The following rule R₁ is a special case of a disjunctive rule of refutation (the last kind discussed above, T9c).

+R24-1. $V^\circ \xrightarrow{\text{ad}} \Lambda^\vee$.

This rule does not refer to any particular form of sentences, and therefore it is possible to use it in connection with any calculus whatever. It turns out that, for many calculi, the addition of this rule is useful. This is the case if a calculus K contains sentences which, though false in the interpretation intended for K , are not C-false in K but only C-comprehensive (D₁, corresponding to [I] D30-6). By adding R₁, these sentences become C-false and hence false in every true interpretation (T18c, f).

D24-1. \mathfrak{X}_i is **C-comprehensive** in $K =_{\text{Df}} \mathfrak{X}_i \xrightarrow{\text{c}} \text{every sentence in } K$.

T24-11. \mathfrak{X}_i is C-comprehensive (in K) if and only if $\mathfrak{X}_i \xrightarrow{\text{c}} V^\circ$. (From D₁, T23-23.)

T24-12. If \mathfrak{X}_i is C-false (in K), it is C-comprehensive. (From D₁, T23-28.)

T24-13. In any calculus, the following junctives are C-comprehensive:

a. V^* . (From T11, T23-5.)

b. Λ^* . (From D1, T23-26.)

T24-14. If V^* is C-false in K , then every C-comprehensive junctive in K is C-false. (From T11, T23-22.)

+**T24-18.** If K' is constructed out of K by adding the rule R1, then the assertions (a), (b), (c) hold. If, moreover, S is a true interpretation for K' containing no other sentences than K' , then, in addition, the assertions (d), (e), (f) hold. (V is the universal sentential class in K , in K' , and in S .)

a. $V^* \xrightarrow{C} \Lambda^*$ in K' .

b. V^* is C-false in K' .

c. \mathfrak{X}_i is C-false in K' if and only if \mathfrak{X}_i is C-comprehensive in K .

d. V^* is false in S .

+e. There is at least one false sentence in S .

f. If \mathfrak{X}_i is C-comprehensive in K , it is false in S .

Proof. a. From T23-3. — b. From (a), D23-6. — c. If \mathfrak{X}_i is C-false in K' , it is C-comprehensive in K' (T12) and hence in K . (The class of C-comprehensive junctives is not increased by R1, because the two junctives involved are C-comprehensive anyway; see T13 a,b.) If \mathfrak{X}_i is C-comprehensive in K , it is C-comprehensive in K' and, hence, C-false in K' ((b), T14). — d. From (b), T23-17. — e. From (d), T21-19 b. — f. If \mathfrak{X}_i is C-comprehensive in K , it is C-false in K' (c), and hence false in S (T23-17).

T18b and e show that the effect of R1 in a calculus with junctives is the same as that of the rule of refutation without a disjunctive: “ V is directly C-false” (R20-1) in a calculus of the ordinary kind. There is, however, a difference between the two rules. R24-1 is part of the definition of ‘direct C-implicate’ and does not involve ‘directly C-false’ as an additional basic concept, as R20-1 does.

We shall apply R1 in order to supplement PC (§ 26). This will exclude one kind of non-normal interpretation of PC.

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The junctives introduced in the preceding chapter are here used for the construction of a new system of propositional logic (§ 25) and a new propositional calculus, called PC* (§ 26). PC*, in contradistinction to PC, is a full formalization of propositional logic (§ 27).

§ 25. Junctives in Propositional Logic

On the basis of the systems of radical and L-concepts for junctives (§§ 21 and 22), the previous system of propositional logic (§§ 11 to 13) is adapted to junctives. Among the results: a conjunction_L is L-equivalent with the conjunctive of the components (T3b), a disjunction_L with the disjunctive (T4b).

In the last chapter the junctives were introduced, and their use in semantics and syntax was discussed in general. Now we are coming back to propositional logic and propositional calculus, in order to find out what changes these systems undergo if junctives are used.

As a system of semantical concepts for junctives, we shall use that discussed at the end of § 22, based on D22-1 and 2. As explained in § 22, this system comprehends all theorems of § 22 (including P22-1 to 15, regarded as theorems) and of § 21; further, it comprehends the general definitions and theorems of § 11 (up to D11-12) and of [I] § 18 modified by replacing any reference to a sentential class \mathfrak{R}_i by a reference to the corresponding conjunctive \mathfrak{R}_i^* . Then we add, as a system of propositional logic based on NTT, the pertinent definitions and theorems in § 11 (from D11-14 on), § 12, and § 13, with the same modification. On this basis, we shall state here a few more theorems concerning connectives of NTT and junctives.

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+**T25-3a [b]**. If S contains a sign of conjunction $[_L]$, and \mathcal{S}_i and \mathcal{S}_j are closed, then $\{\mathcal{S}_i, \mathcal{S}_j\}^*$ is $[L]$ -equivalent to $\text{con}_{[L]}(\mathcal{S}_i, \mathcal{S}_j)$. (Corresponds to T13-14.)

+**T25-4a [b]**. If S contains a sign of disjunction $[_L]$, and \mathcal{S}_i and \mathcal{S}_j are closed, then $\{\mathcal{S}_i, \mathcal{S}_j\}^v$ is $[L]$ -equivalent to $\text{dis}_{[L]}(\mathcal{S}_i, \mathcal{S}_j)$. (From T13-13(2), D22-2, T11-6(2).)

Analogous theorems hold for conjunctives and disjunctives with any finite number of elements.

T25-7a [b]. If S contains a sign of negation $[_L]$, then, for any closed \mathcal{S}_i , $\{\mathcal{S}_i, \text{neg}_{[L]}(\mathcal{S}_i)\}^*$ is $[L]$ -false. (From T3, T13-28(1).)

T25-8a [b]. If S contains a sign of negation $[_L]$, then, for any closed \mathcal{S}_i , $\{\mathcal{S}_i, \text{neg}_{[L]}(\mathcal{S}_i)\}^v$ is $[L]$ -true. (From T4, T13-25(1).)

§ 26. The Calculus PC*

The calculus PC_1^* with junctives (D1) is constructed out of PC_1 (D2-2) by adding two rules, a disjunctive rule of inference (6) and a disjunctive rule of refutation (7). PC_1^{*D} (D2), in analogy to PC_1^D (D3-6), contains definitions for the other connectives. The general concept of forms of PC^* is defined (D3).

Now we shall make use of disjunctive rules in order to supplement PC so as to exclude the possibility of non-normal interpretations for the connectives. We call the resulting calculus PC^* . It will be shown (§ 27) that this calculus fulfills the purpose.

The first kind of non-normal interpretation (T16-6) is such that all sentences become true, even those which are C-comprehensive and hence L-false in the intended (L-normal) interpretation. We have seen that, if any calculus possesses this unwanted feature, it can be removed by the addition of R24-1 (see T24-18e, f). This is rule (7) in PC_1^* (D1 below).

The second kind of non-normal interpretation (Tr6-7) is such that rule Dj4 of NTT (§ 10) is violated. This means that PC does not exclude a true interpretation in which two closed sentences \mathcal{S}_i and \mathcal{S}_j are false but, nevertheless, their disjunction_C sentence $\text{dis}_C(\mathcal{S}_i, \mathcal{S}_j)$ is true. On the other hand, in the system of propositional logic on the basis of NTT, if a disjunction of closed sentences is true, at least one of the two components is true, and hence their disjunctive is true; in other words, $\text{dis}(\mathcal{S}_i, \mathcal{S}_j) \rightarrow \{\mathcal{S}_i, \mathcal{S}_j\}^V$. In order to ensure that this should be the case in every true interpretation, we have merely to add a corresponding disjunctive rule: " $\text{dis}_C(\mathcal{S}_i, \mathcal{S}_j) \xrightarrow{\text{dC}} \{\mathcal{S}_i, \mathcal{S}_j\}^V$, for closed \mathcal{S}_i and \mathcal{S}_j ". This is rule (6) in PC_1^* (D1).

As an example of a form of PC, we have previously stated PC_1 (D2-2). The corresponding form PC_1^* consists of the same rules (with conjunctives instead of classes) and, in addition, the two disjunctive rules (6) and (7) just mentioned.

+D26-1. K contains PC_1^* with neg_C as sign of negation_C and dis_C as sign of disjunction_C =_{DI} K fulfills the following conditions:

- a. neg_C is a singulary and dis_C a binary general connective in K .
- b. The relation of direct C-implication holds in the following cases for any $\mathcal{S}_i, \mathcal{S}_j$, and \mathcal{S}_k (but not necessarily only in these cases):
 - 1, 2, 3, 4, as in D2-2b but with ' Λ ' instead of ' Λ '.
 5. $\{\mathcal{S}_i, \text{dis}_C(\text{neg}_C(\mathcal{S}_i), \mathcal{S}_j)\}^* \xrightarrow{\text{dC}} \mathcal{S}_j$.
 6. $\text{dis}_C(\mathcal{S}_i, \mathcal{S}_j) \xrightarrow{\text{dC}} \{\mathcal{S}_i, \mathcal{S}_j\}^V$, where \mathcal{S}_i and \mathcal{S}_j are closed.
 7. $V^* \xrightarrow{\text{dC}} \Lambda^V$.

In analogy to ' PC_1^D ' (D3-6), we define ' PC_1^{*D} '.

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+D26-2. K contains $PC_1^{*D} =_{Df}$ K contains PC_1^* and, in addition, definition rules on the basis of neg_c and dis_c for signs for all other singulary and binary connections_c, with definitia as given in column (5) of the table in § 3.

The general concept of forms of PC^* (D3) is defined in analogy to D4-1.

D26-3. A calculus K_p contains (a form of) $PC^* =_{Df}$ there are calculi K_m and K_n such that the following conditions are fulfilled:

- a. K_m contains PC_1^{*D} .
- b. K_n is a conservative sub-calculus of K_m ([I] D31-7).
- c. For every sentence \mathfrak{S}_i in K_m there is a sentence \mathfrak{S}_j in K_n (and K_m) which is C-equivalent to \mathfrak{S}_i in K_m .
- d. K_p is isomorphic to K_n by a correlation H .

§ 27. PC^* is a Full Formalization of Propositional Logic

The interpretations for PC^* are examined. It is found that the connectives neg_c and dis_c in PC_1^* have a normal interpretation in any true interpretation for PC_1^* and an L-normal interpretation in any L-true interpretation for PC_1^* (T1). The same holds for all 4 + 16 connectives in PC_1^{*D} (T5), and likewise in any other form of PC^* (T9). Thus PC^* is a full formalization of propositional logic.

Now it will be shown that the connectives in PC^* can only be interpreted normally.

+T27-1a [b]. If K contains PC_1^* and S is an [L]-true interpretation for K containing only the sentences of K , then neg_c and dis_c in K have an [L]-normal interpretation in S .

Proof. Let K_m be a calculus of the ordinary kind (without rules of refutation) corresponding to K (i.e. the relation between K_m and K is that described for K_m and K_n in T23-40 A, B, C; D is fulfilled

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vacuously). Then, K_m contains PC_1 . Therefore, analogues to all previous theorems concerning PC_1 hold for K (T23-41). — I, for (a) (see remark preceding T15-4). Let \mathcal{S}_i and \mathcal{S}_j be any closed sentences in K and S . $\text{disc}(\mathcal{S}_i, \mathcal{S}_j) \xrightarrow{\text{dC}} \{\mathcal{S}_i, \mathcal{S}_j\}^V$ in K (D26-1b(6)); therefore $\text{disc}(\mathcal{S}_i, \mathcal{S}_j) \rightarrow \{\mathcal{S}_i, \mathcal{S}_j\}^V$ in S (D23-1a). Hence, if $\text{disc}(\mathcal{S}_i, \mathcal{S}_j)$ is true in S , $\{\mathcal{S}_i, \mathcal{S}_j\}^V$ is also true, and at least one of the two sentences is true (D21-4). In other words, if both sentences are false, $\text{disc}(\mathcal{S}_i, \mathcal{S}_j)$ is false. Thus disc generally satisfies rule Dj4 in NTT. Further, disc generally satisfies Dj1 to 3 (T15-4a). Hence, disc is a sign of disjunction in S (T11-12a) and has a normal interpretation (D15-1a). neg_C in K cannot violate N2 in S because otherwise disc would violate Dj4 (T15-7d). In consequence of rule (7) (D26-1b(7)), at least one sentence in S is false (T24-18e). Therefore, neg_C generally satisfies N1 (T15-6), is a sign of negation in S , and has a normal interpretation. — II, for a [b]. Let \mathcal{S}_k be $\text{disc}(\mathcal{S}_i, \mathcal{S}_j)$. Because of rule (6), $\mathcal{S}_k \xrightarrow{\text{dC}} \{\mathcal{S}_i, \mathcal{S}_j\}^V$ in S (D23-1). Therefore, $\neg R_k + R_i + R_j$ contains rs (D22-2, [I] T18-28) [is V_* (D22-2, [I] T18-10)]. Hence, disc generally [L-]satisfies Dj4 (T13-10(4)), and likewise Dj1, 2, and 3 (T15-4). Hence, disc is a sign of disjunction_[L] in S (T11-12). Let \mathcal{S}_n be $\text{neg}_C(\mathcal{S}_i)$, and \mathcal{S}_p be $\text{disc}(\mathcal{S}_i, \mathcal{S}_n)$. Then, because of rule (6) as above, $\neg R_p + R_i + R_n$ contains rs [is V_*]. \mathcal{S}_p is C-true in K (T5-1a), and hence [L-]true in S (T23-16 [T23-19a]). Hence R_p contains rs [is V_*]. Therefore, $R_i + R_n$ contains rs [is V_*]. Let \mathcal{R}_m^* be $\{\mathcal{S}_i, \mathcal{S}_n\}^*$. $\mathcal{R}_m^* \xrightarrow{\text{c}} V^*$ in K (T5-21, T23-23); hence $\mathcal{R}_m^* \xrightarrow{\text{dC}} V^*$ in S (T23-11). Because of rule (7), V^* is [L-]false in S (T24-18d [T22-25]). Therefore, \mathcal{R}_m^* is [L-]false ([I] T9-11 [P22-7]). Hence, $R_i \times R_n$ does not contain rs [is Λ_*] (D22-1); hence, $\neg R_i + (\neg R_n)$ contains rs [is V_*]. We found previously the same for $R_i + R_n$. Therefore, neg_C is a sign of negation_[L] in S (T13-5).

Note on Tr. If K contains two sub-calculi of the form PC_1^* , and S is an [L-]true interpretation for K , then obviously, according to T1, both signs of negation_C and both signs of disjunction_C have an [L-]normal interpretation in S . But the same result is obtained if only one of the sub-calculi has the form PC_1^* while the other has the ordinary form PC_1 without disjunctive rules, provided that K fulfills the conditions (B) and (C) in T6-10. This follows from T15-1 and 2.

+T27-5a [b]. If K contains PC_1^{*D} and S is an [L-]true interpretation for K containing only the sentences of K ,

then all connectives of PC_1^{*D} in K have an $[L]$ -normal interpretation in S . (From T_1 , T_{16-2} . Analogues to previous theorems concerning PC_1^D hold here; see proof for T_1 .)

+**T27-9a** $[b]$. Let K contain any form of PC^* . If S is an $[L]$ -true interpretation for K containing only the sentences of K , then every connective of PC^* in K has an $[L]$ -normal interpretation in S .

Proof for a $[b]$. Let the conditions be fulfilled, and K_p be K . Then there are calculi K_m and K_n fulfilling the conditions (a) to (d) in D26-3. Let us further make the following assumptions. (They do not restrict the generality, except that we refer only to a binary connective; the consideration for a singular connective would be analogous.) A. Let S_p be S ; hence S_p is an $[L]$ -true interpretation for K_p . B. Let a_n be a sign in K_n and hence in K_m , and let it be the sign for $cConn_2^2$ in PC_1^{*D} in K_m . C. Let \mathfrak{S}_n and \mathfrak{S}'_n be any closed sentences in K_n . D. Let a_p , \mathfrak{S}_p , and \mathfrak{S}'_p be the H -correlates (according to D26-3d) in K_p to a_n , \mathfrak{S}_n , and \mathfrak{S}'_n respectively. E. Let the system S_n be constructed in such a way that the following conditions (a) and (b) hold; then (c) holds too. a. S_n contains the same sentences as K_n . b. The truth-condition stated by the rules of S_n for any sentence in K_n is the same as that stated by the rules of S_p for the H -correlate of that sentence in K_p . c. Any sentence in S_n is equivalent and even L -equivalent to its H -correlate in S_p . (Concerning the application of semantical relations to items in different systems, see remarks at the end of $[I]$ § 12 and of $[I]$ § 16.) F. Let the system S_m be constructed in such a way that the following conditions hold: a. S_m contains the same sentences as K_m . b. For any sentence in S_m which belongs also to S_n , the same truth-condition is laid down in S_m as in S_n . c. For any sentence in S_m which does not belong to S_n , the same truth-condition is laid down in S_m as for a sentence in S_n which we choose arbitrarily among those which are C -equivalent to it in K_m (D26-3c); hence the two sentences are L -equivalent in S_m .

On the basis of these assumptions, the following holds:

1. a_p is a sign for $cConn_2^2$ in K_p . (From B, D, D4-3.) — 2. S_n is an $[L]$ -true interpretation for K_n . (From A, D26-3d, E.) — 3. If a sentence in K_m is C -equivalent in K_m to each of several sentences in K_n , then these sentences are C -equivalent to one another in K_m and hence also in K_n (D26-3b), and $[L]$ -equivalent to one another in S_n

(2) and hence also in S_m (F(b)). — 4. If a sentence \mathcal{S}_i in K_m which does not belong to K_n is C-equivalent in K_m to a sentence \mathcal{S}_j in K_n , then there is a sentence \mathcal{S}_k in K_n (namely, that chosen according to F(c)) which is [L-]equivalent to \mathcal{S}_i (F(c)) and to \mathcal{S}_j in S_m (3); hence \mathcal{S}_i is [L-]equivalent to \mathcal{S}_j in S_m . — 5. If $\mathcal{T}_i \xrightarrow{C} \mathcal{T}_j$ in K_m , then $\mathcal{T}_i \xrightarrow{[L]} \mathcal{T}_j$ in S_m . (Proof. There are \mathcal{T}_k and \mathcal{T}_l in K_n such that \mathcal{T}_i is C-equivalent to \mathcal{T}_k in K_m (D26-3c) and likewise \mathcal{T}_j to \mathcal{T}_l . Then \mathcal{T}_i is [L-]equivalent to \mathcal{T}_k in S_m (4), and likewise \mathcal{T}_j to \mathcal{T}_l . Since $\mathcal{T}_i \xrightarrow{C} \mathcal{T}_j$ in K_m , $\mathcal{T}_k \xrightarrow{C} \mathcal{T}_l$ in K_m (T23-4), and hence also in K_n (D26-3b). Therefore $\mathcal{T}_k \xrightarrow{[L]} \mathcal{T}_l$ in S_n (2); hence $\mathcal{T}_i \xrightarrow{[L]} \mathcal{T}_j$ in S_m ([I] T9-14b [P22-5]).) — 6. S_m is an [L-]true interpretation for K_m . (From D23-1, T23-3, (5).) — 7. a_n is a sign for $_{[L]}Conn^2$ in S_m (from D26-3a, F(a), (6), B), and hence in S_n (F(b)). — 8. $a_n(\mathcal{S}_n, \mathcal{S}'_n)$ in S_n is [L-]equivalent to $a_p(\mathcal{S}_p, \mathcal{S}'_p)$ in S_p . (From D, E(c).) — 9. a_p has the same [L-]characteristic in S_p as a_n in S_n . (From E(c), (8).) — 10. a_p is a sign for $_{[L]}Conn^2$ in S_p . (From (9), (7).) — 11. a_p in $K_p (= K)$ has an [L-]normal interpretation in $S_p (= S)$. (From D15-1, (1), (10).)

It has previously been explained (at the end of § 18) under what condition a calculus may be called a *full formalization of propositional logic* as represented by the rules NTT. T1, 5, and 9 show that any calculus containing the special forms PC_1^* or PC_1^{*D} or in general any form of PC^* fulfills that condition.

We have formerly seen (§ 19 at the end) that the following two principles hold in the propositional logic, but that their validity is not assured by PC:

A. Principle of (Excluded) Contradiction. A sentence and its negation cannot both be true.

B. Principle of Excluded Middle. A sentence and its negation cannot both be false.

It follows from the preceding results that the validity of both principles is assured by PC^* ; that is to say, the principles hold in any true interpretation of a calculus containing PC^* with respect to neg_C in PC^* . In the case of PC_1^* , this

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can easily be seen directly on the basis of the rules (see D26-1).

Let K contain PC_1^* and hence PC_1 . Let S be a true interpretation for K , and \mathcal{S}_i be any closed sentence in K . Then every sentence, and hence also V , is a C-implicate of $\{\mathcal{S}_i, \text{neg}_C(\mathcal{S}_i)\}$ in PC_1 (T5-21). Therefore, V^* , and hence, according to rule (7) (D26-1), Λ^v is a C-implicate of $\{\mathcal{S}_i, \text{neg}_C(\mathcal{S}_i)\}^*$ in K . Hence, this conjunctive is false in S (T23-11a, T21-16), and at least one of the two sentences \mathcal{S}_i and $\text{neg}_C(\mathcal{S}_i)$ is false in S (T21-1). This is A. Further, $\text{disc}(\mathcal{S}_i, \text{neg}_C(\mathcal{S}_i))$ is C-true in K (T5-1a). Hence, according to rule (6), $\{\mathcal{S}_i, \text{neg}_C(\mathcal{S}_i)\}^v$ is C-true in K and true in S . Therefore at least one of the sentences \mathcal{S}_i and $\text{neg}_C(\mathcal{S}_i)$ is true in S (D21-4). This is B.

F. FULL FORMALIZATION OF FUNCTIONAL LOGIC

The problem of the possibility of a full formalization of functional logic (with respect to predicates of first level only) is discussed. The ordinary form FC_1 of the (lower) functional calculus is not sufficient for this purpose (§ 28). With the help of transfinite junctives (§ 29), the calculus FC_1^* is constructed (§ 30). This calculus is a full formalization of functional logic (§ 31). Finally, an alternative to the use of junctives is explained, based on a concept called 'involution'; with its help, a calculus FC_1^{**} is constructed, which is likewise a full formalization of functional logic (§ 32).

§ 28. The Functional Calculus (FC)

As logic of functions, we take a system with predicates of first level and a denumerable set of individuals, all of them designated by individual constants. The rules of a special form (FC_1) of the ordinary lower functional calculus (FC) are laid down (D_1 and 2). The concepts of normal interpretations for the universal and the existential operators are defined (D_6 and 7). A true interpretation of FC_1 is indicated in which the operators have a non-normal interpretation. Therefore, FC is not a full formalization of functional logic. — The result of substituting an individual constant for a free individual variable in \mathfrak{S}_i is called an instance of \mathfrak{S}_i (D_3).

In the previous chapters we have studied the ordinary propositional calculus PC. By using junctives, we have transformed it into a new calculus PC^* , which is a full formalization of propositional logic. Analogously, we shall now study the ordinary functional calculus FC, and transform it into a new calculus FC^* , which is a full formalization of functional logic. Here the use of transfinite junctives will be necessary.

For the sake of simplicity and brevity we shall restrict the

following investigation in several respects. We shall analyze only the lower functional calculus (containing predicates of the first level only). We shall discuss only one form of it; we shall call it FC_1 because it is analogous to the form PC_1 of PC (D2-2). As in the case of PC, the results found for FC_1 hold in an analogous way for the other forms of FC also.

The form FC_1 to be explained below is, in its essential features, the form constructed by Hilbert and Bernays; see Hilbert and Ackermann [Logik], Kap. III, and Hilbert and Bernays [Grundl. Math. I], § 4. We simplify this form here by using individual variables as the only variables. We discard propositional variables (see § 2) and predicate variables; instead of Hilbert's primitive sentences we have then to use primitive sentential schemata (as in PC_1 ; see § 2). The inclusion of these two kinds of variables would not, however, cause any difficulty in establishing a full formalization.

We presuppose a system of functional logic; the task will be to give a formalization, and if possible a full formalization, of this functional logic. We suppose that the realm of individuals, i.e. the realm of values for the individual variables, is denumerable (i.e. there is a one-one correlation between the individuals and the natural numbers). Further, we presuppose that every individual is designated by an individual constant in the system (e.g. 'a', 'b', . . .). [Instead of individual constants, individual expressions might be used, as e.g. the so-called accented expressions 'o', 'o'', 'o'''', etc., as used by Hilbert in another system, and in [Syntax] in languages I and II; see [Syntax] § 3.]

The calculus FC is a calculus of the ordinary kind; that is to say, we deal here not with junctives but with sentences and sentential classes only. Later we shall again make use of junctives.

The Calculus FC_1

1. Classification of signs

- a. Sign of negation (neg_C), sign of disjunction (dis_C), parentheses, comma (as in PC_1).
- b. Individual constants.
- c. Individual variables (i).
(b) and (c) are called individual signs (in).
- d. Any number of predicates of any degree.
(pr^n is the class of predicates of degree n .)
- e. The existential sign ' \exists ' (its name in the metalanguage is also ' \exists ').

2. Rules of formation

+D28-1. An expression in FC_1 is a *sentence* in $FC_1 =_{D1}$ it has one of the following forms (a) to (e).

- a. $\text{pr}^n(\text{in}_{k1}, \text{in}_{k2}, \text{in}_{k3}, \dots, \text{in}_{kn})$ (an atomic sentence consisting of a predicate of degree n with n individual signs as arguments).
- b. $\text{neg}_C(\mathcal{S}_i)$.
- c. $\text{dis}_C(\mathcal{S}_i, \mathcal{S}_j)$.
- d. $(i_k)(\mathcal{S}_i)$.
- e. $(\exists i_k)(\mathcal{S}_i)$.

3. Rules of deduction

+D28-2. **Direct C-implication in FC_1** holds in the following cases (1) to (5), (8) to (13), and only in these.

- 1 to 5, as in PC_1 ; see D2-2.
8. $\Lambda \xrightarrow{dC} \text{dis}_C(\text{neg}_C((i_k)(\mathcal{S}_i)), \mathcal{S}_i)$.
9. $\Lambda \xrightarrow{dC} \text{dis}_C(\text{neg}_C(\mathcal{S}_i), (\exists i_k)(\mathcal{S}_i))$.
10. $\mathcal{S}_i \xrightarrow{dC} \mathcal{S}_i \left(\text{in}_m^{i_k} \right)$, where in_m is not a variable which would be bound at one of the places of substitution after the substitution.

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11. $\text{disc}(\mathcal{S}_i, \mathcal{S}_j) \xrightarrow{\text{dc}} \text{disc}(\mathcal{S}_i, (i_m)(\mathcal{S}_j))$, provided i_m does not occur as a free variable in \mathcal{S}_i .
12. $\text{disc}(\text{neg}_c(\mathcal{S}_i), \mathcal{S}_j) \xrightarrow{\text{dc}} \text{disc}(\text{neg}_c((\mathcal{A}i_m)(\mathcal{S}_i)), \mathcal{S}_j)$, provided i_m does not occur as a free variable in \mathcal{S}_j .
13. $\mathcal{S}_i \xrightarrow{\text{dc}} \mathcal{S}_j$ where \mathcal{S}_i contains as part a sentence \mathcal{S}_k of the form $(i_m)(\mathcal{S}_p)$ or $(\mathcal{A}i_m)(\mathcal{S}_p)$, and \mathcal{S}_j is constructed out of \mathcal{S}_i by replacing \mathcal{S}_k with $(i_n)(\mathcal{S}_p(i_n^{i_m}))$ or $(\mathcal{A}i_n)(\mathcal{S}_p(i_n^{i_m}))$ respectively; here, i_n may be any variable not occurring in \mathcal{S}_p .

Explanation. (8) and (9) correspond to the following two primitive sentences in Hilbert's system, written with a predicate variable ' F ' and with ' \supset ' as a defined sign of implication: ' $(x)(F(x)) \supset F(x)$ ' and ' $F(x) \supset (\mathcal{A}x)(F(x))$ '. (10) is the rule of substitution. $\mathcal{S}_i(i_n^{i_m})$ is the sentence constructed out of \mathcal{S}_i by substituting i_n for i_k at all places where i_k occurs as a free variable in \mathcal{S}_i . (11) and (12) are the rules of insertion for the universal and the existential operator. (13) is the rule for replacing one bound variable by another.

D28-3. \mathcal{S}_j is an **instance** of \mathcal{S}_i with respect to i_k in FC_1 (or in FC_1^* , § 30) =_{df} \mathcal{S}_j has the form $\mathcal{S}_i(i_n^{i_k})$ where i_n is an individual constant.

Examples. ' $P(a)$ ', ' $P(b)$ ', etc., are instances of ' $P(x)$ ' with respect to ' x '. Of ' $P(x) \vee Q(y)$ ', ' $P(a) \vee Q(y)$ ' is an instance with respect to ' x ', ' $P(x) \vee Q(a)$ ' with respect to ' y '. If i_k does not occur as a free variable in \mathcal{S}_i (e.g. ' x ' in ' $R(a,b)$ ', ' $R(a,y)$ ', ' $(x)R(a,x)$ '), then \mathcal{S}_i itself is the only instance of \mathcal{S}_i .

D28-4. $\{\mathcal{S}_i(i_n^{i_k})\}$ (in FC_1 or FC_1^*) =_{df} the class of the instances of \mathcal{S}_i with respect to i_k .

In T4, we list some examples for C-implication in FC_1 , for reference in subsequent proofs.

T28-4. In each of the following cases, \mathcal{S}_j is a *C-implicate* of \mathcal{S}_i in FC_1 .

\mathcal{S}_i is:		\mathcal{S}_j is:
a.	$(i_k)(\mathcal{S}_p)$	$\mathcal{S}_p(i_k)$
b.	\mathcal{S}_p	$(i_k)(\mathcal{S}_p)$
c.	$\mathcal{S}_p(i_k)$	$(\exists i_k)(\mathcal{S}_p)$
d.	$(\exists i_k)(\mathcal{S}_p)$	$\text{neg}_C((i_k)(\text{neg}_C(\mathcal{S}_p)))$
e.	(d) in inverse order	
f.	$(i_k)(\mathcal{S}_p)$	$\text{neg}_C((\exists i_k)(\text{neg}_C(\mathcal{S}_p)))$
g.	(f) in inverse order	

Proofs. In each of the cases described, $\text{disc}(\text{neg}_C(\mathcal{S}_i), \mathcal{S}_j)$ is known to be provable in FC_1 (see e.g. Hilbert [Logik], Kap. iii, § 6, or, for a slightly different calculus, Whitehead and Russell [Princ. Math.], vol. I, *10), and is hence C-true in FC_1 ([I] T29-100). Therefore, since FC_1 contains PC_1 , $\mathcal{S}_i \xrightarrow{C} \mathcal{S}_j$ (T7-1).

Let S be a semantical system containing individual variables and individual constants; S may, for instance, contain the signs and sentences of FC_1 . The values ([I] § 11) of the individual variables are called the **individuals** in S . We presuppose here that all individual constants in S are value expressions for the individual variables. Therefore, the designata of the individual constants in S belong to the individuals; we call them *directly designated individuals*. (Other individuals in S may either be designated by complex individual expressions, e.g. full expressions of functors, or not be designated at all in S .) Analogously, we call those attributes which are designated by predicates in S *directly designated attributes* in S .

By the normal interpretation of FC we mean that which is ordinarily used. According to it, ' $(x)(P(x))$ ' means 'for every x , x is P (i.e. x has the property P)', and ' $(\exists x)(P(x))$ ' means 'for at least one x , x is P '. Hence, if the operators have a normal interpretation in S , then ' $(x)(P(x))$ ' is true in S if and only if every individual in S has the property determined ([I] § 11) by the sentential function ' $P(x)$ '; and

' $(\exists x)(P(x))$ ' is true if and only if at least one individual has that property. This consideration leads to the following definitions D6 and 7.

+D28-6. The **universal operator** in FC_1 has a **normal interpretation** in $S =_{Df} S$ is a true interpretation for FC_1 , and any closed sentence of the form $(i_k)(\mathfrak{S}_p)$ is true in S if and only if every individual in S has the property determined by \mathfrak{S}_p .

+D28-7. The **existential operator** in FC_1 has a **normal interpretation** in $S =_{Df} S$ is a true interpretation for FC_1 , and any closed sentence of the form $(\exists i_k)(\mathfrak{S}_p)$ is true in S if and only if at least one individual in S has the property determined by \mathfrak{S}_p .

It is easy to see that there are true interpretations for FC_1 in which the operators have a non-normal interpretation, even if the connectives have a normal interpretation. Thus, e.g., S_1 may be a true interpretation of such a kind that the connectives have a normal interpretation in S_1 , while ' $(x)P(x)$ ' is interpreted in S_1 as "every individual is P, and b is Q", and ' $(\exists x)P(x)$ ' as "at least one individual is P, or b is not Q". (For this example, see [Syntax] § 62.) Therefore FC_1 is not a full formalization of the logic of functions.

The rest of this section is of less importance; the results will not be used in the subsequent sections. Tro shows that some of the previous theorems which contain the condition of extensibility (D6-1) hold also for FC_1 and hence for many other calculi constructed on the basis of FC_1 .

T28-10. The rules of inference in FC_1 (D2(5), (10) to (13)) are extensible.

Proof. For (5): T6-5. — The proof for (10) is analogous to that for T6-3a. — For (11). From $\text{disc}(\mathfrak{S}_k, \text{disc}(\mathfrak{S}_i, \mathfrak{S}_j))$, C-implication leads, step for step, to the following sentences, under the conditions required for i_m (in rule (11)) and for \mathfrak{S}_k (in D6-1): $\text{disc}(\text{disc}(\mathfrak{S}_k, \mathfrak{S}_i), \mathfrak{S}_j)$ (T5-3k); $\text{disc}(\text{disc}(\mathfrak{S}_k, \mathfrak{S}_i), (i_m)(\mathfrak{S}_j))$ (rule (11)); $\text{disc}(\mathfrak{S}_k, \text{disc}(\mathfrak{S}_i, (i_m)(\mathfrak{S}_j)))$

(T5-3k). — For (12). From $\text{disc}(\mathcal{S}_k, \text{disc}(\text{neg}_C(\mathcal{S}_i), \mathcal{S}_j))$, C-implication leads to the following sentences, under the conditions required for i_m and \mathcal{S}_k : $\text{disc}(\text{neg}_C(\mathcal{S}_i), \text{disc}(\mathcal{S}_k, \mathcal{S}_j))$ (T5-3j); $\text{disc}(\text{neg}_C((\exists i_m)(\mathcal{S}_i)), \text{disc}(\mathcal{S}_k, \mathcal{S}_j))$ (rule (12)); $\text{disc}(\mathcal{S}_k, \text{disc}(\text{neg}_C((\exists i_m)(\mathcal{S}_i)), \mathcal{S}_j))$ (T5-3j). — For (13). Between the disjunction_C sentences, direct C-implication holds, by rule (13) itself.

If K contains a form of FC with predicate variables and contains a rule of simple substitution for predicate variables and a rule of substitution with arguments for predicate variables, then these rules can easily be shown to be extensible. The proof is analogous to that for T6-3a.

Let us consider a calculus K containing the following rule (11'), which is simpler but weaker than (11). Rule (11'): $\mathcal{S}_i \xrightarrow{\text{adC}} (i_m)(\mathcal{S}_i)$. This rule is not necessarily extensible. It is so if K permits the operation known as "shifting the universal operator", i.e. if $(i_m)(\text{disc}(\mathcal{S}_i, \mathcal{S}_j)) \xrightarrow{\text{C}} \text{disc}(\mathcal{S}_i, (i_m)(\mathcal{S}_j))$ in K provided that i_m does not occur as a free variable in \mathcal{S}_i . This is, for instance, the case in the calculus called language II in [Syntax], because of PS II 19 ([Syntax] § 30). Therefore the rule R II 2 ([Syntax] § 31), which corresponds to rule (11') above, is extensible, as is shown by [Syntax] Theorem 32.2a. Hence T10 holds also for language II.

The reason for the restriction with respect to free variables in the definition for 'extensible' (D6-1) can now be explained by an example in FC. If we take the rule (11') just mentioned, then ' $P(x)$ ' $\xrightarrow{\text{adC}}$ ' $(x)P(x)$ '. On the other hand, ' $\sim P(x) \vee (x)P(x)$ ' (\mathcal{S}_2) is certainly not a C-implicate of ' $\sim P(x) \vee P(x)$ ' (\mathcal{S}_1), because \mathcal{S}_1 is C-true while \mathcal{S}_2 is C-equivalent to ' $(x)(\sim P(x) \vee (x)P(x))$ ' and hence to ' $(x)(\sim P(x) \vee (x)P(x))$ ' and is therefore C-indeterminate. (In the normal interpretation, \mathcal{S}_2 is false if some individuals are P and some are not.) This shows that the restriction in D6-1 is necessary. On the other hand, it can be shown that the restriction is strong enough. It suffices to require that any free variables in \mathcal{S}_k , i.e. the component added, do not occur freely in the rest, without requiring that \mathcal{S}_k be closed; because ' $P(y) \vee Q(x)$ ' (where ' $P(y)$ ' takes the place of \mathcal{S}_k) is C-equivalent to ' $(y)(P(y)) \vee Q(x)$ '.

Earlier (at the end of § 6), a procedure was indicated for transforming a non-extensible rule into an extensible one. As an example, let us suppose that a calculus K contains rule (11') in such a way that (11') is not extensible, e.g. by containing only the rules of deduction

of FC_1 (D_2) but with (11') instead of (11). Then the procedure described earlier would transform (11') into the extensible rule (11).

+T28-11. If K contains FC_1 and there are no other rules of inference in K than those of FC_1 ($D_2(5), (10)$ to (13)), then the assertions (a) and (b) in T6-14 hold for K . (From T10, T6-10, T6-12.)

T11 may be called the *deduction theorem for FC_1* (see remark on T6-12). T10 holds also for the other forms of FC and for the customary forms of the higher functional calculus. Therefore T11 holds for very many calculi in practical use. Many postulate systems are constructed on the basis of the (lower or higher) functional calculus; the postulates (axioms) are additional primitive sentences (see [Foundations] § 16); in most cases there are no additional rules of inference.

§ 29. Transfinite Junctives

If the rules of deduction defining the concept of direct C-implication (or direct derivability) are such that in any given case we can find out by a finite number of steps whether or not that concept holds, then that concept and those rules are called definite; otherwise, indefinite. An indefinite rule usually refers to a transfinite junctive. This is, in the cases of indefinite rules used by logicians so far, a transfinite sentential class (or conjunctive) as C-implicans. But it is also possible to use a disjunctive rule with a transfinite disjunctive as C-implicate. The use of indefinite rules referring to transfinite junctives will be necessary for solving the task of a full formalization of functional logic.

In this section we shall discuss indefinite rules and transfinite junctives because we shall later find them necessary for the construction of a calculus which is to be a full formalization of functional logic (§ 30).

A concept is called **definite** (or effective) if its definition provides a so-called method of decision (*Entscheidungsverfahren*), i.e. a method by whose application we can decide in

any given case in a finite number of steps whether or not the concept holds ([Syntax] § 15). If a concept is not definite, it is called **indefinite**. If one of the basic concepts defined by the rules of a calculus K (usually 'sentence' and 'directly derivable', including 'primitive sentence'; sometimes also 'directly C-false') is definite, we call the rules defining that concept definite. If all rules (rules of formation and rules of deduction) of K are definite, we call K a **definite calculus**; otherwise, an **indefinite calculus**. All calculi of the customary kind are definite. But indefinite calculi seem to be admissible and convenient and even necessary for certain purposes.

The above remark concerning the concept 'definite' is meant as a rough explanation only. Within an arithmetized syntax (Gödel's method, see [Syntax] § 19) an exact definition can be given. In this method, expressions are correlated with natural numbers; therefore properties and relations of expressions, e.g. the basic concepts of a calculus mentioned above, are correlated with functions of natural numbers. A syntactical concept is definite if the correlated arithmetical function has a certain property for which several exact definitions have been given which have been shown to coincide with one another: 'λ-definable function' (Church and Kleene), 'general recursive function' (Herbrand and Gödel), 'computable function' (Turing; see *Journ. Symb. Log.*, vol. 2, 1937, p. 153).

Concerning indefinite rules which have been used by logicians, see, below, the comment on D30-3 (14). Concerning the question whether indefinite rules are admissible, see [Syntax] §§ 43 and 45. An example of a task which cannot be solved without the use of indefinite rules is that of constructing an L-exhaustive calculus ([I] D36-3) for arithmetic (see [Syntax] §§ 14 and 34a, [Foundations] § 10 at the end).

If indefinite rules of deduction for calculi are admitted, then the rules may refer not only to sentences or finite sentential classes but also to transfinite sentential classes. We call a rule of deduction which refers to a transfinite sentential class or junctive a **transfinite rule**.

All indefinite rules of deduction which logicians have used

so far seem to be transfinite. Most, if not all, are rules of inference (i.e. of the form " \mathfrak{I}_j is directly derivable from \mathfrak{I}_i ") of such a kind that the C-implicans (\mathfrak{I}_i) is a transfinite class while the C-implicate (\mathfrak{I}_j) is a single sentence. The reason for this fact is that the sentential classes have always been taken in the sense which we call now conjunctive, and that the use of a conjunctive is essential only as a C-implicans, not as a C-implicate (§ 23). Now we also use disjunctives; and their occurrence is essential if they are used as C-implicates. Therefore it is now possible to extend the scope of the deductive method still more, by using transfinite rules of a new kind, with a transfinite disjunctive as C-implicate.

Incidentally, in (interpreted, not formalized) logic as represented in L-semantics there are analogous possibilities for the extension of the scope of logical deduction by using a transfinite conjunctive as L-implicans and a transfinite disjunctive as L-implicate.

§ 30. The Calculus FC^*

We construct the calculus FC_1^* (D_3), which is similar to FC_1 ; the difference is that FC_1^* is a calculus with junctives and contains three more rules of deduction. These are the two disjunctive rules which PC_1^* contains in distinction to PC_1 ($D_{26-1(6)}$ and (7)), and a rule ($D_3(14)$) stating that \mathfrak{S}_i is a direct C-implicate of the transfinite conjunctive of the instances of \mathfrak{S}_i . In FC_1^* , a universal sentence is C-equivalent to the conjunctive of the instances of its operand (T_{2c}); an existential sentence is C-equivalent to their disjunctive (T_{3c}).

With the help of junctives, a calculus FC^* can be constructed out of FC such that FC^* represents a full formalization of functional logic. For the sake of brevity, we shall restrict our discussion to the form FC_1^* corresponding to FC_1 . The classification of signs and the rules of formation of FC_1^* are the same as those of FC_1 (D_{28-1}). The rules of deduc-

tion of FC_1^* (see D₃, below) contain those of FC_1 (D₂₈₋₂), with conjunctives instead of sentential classes; further, three rules (6), (7), and (14) are added. The rules (6) and (7) are those which we added earlier to PC_1 in order to construct PC_1^* (see D_{26-1b}(6) and (7)). (14) is a new rule for the universal operator, with a transfinite conjunctive as C-implicans (see D₂₈₋₄). It will be seen later (§ 31) that, on the basis of these rules, not only the universal operator but also the existential operator has a normal interpretation in any true interpretation for FC_1^* .

*Rules of deduction for FC_1^**

+D₃₀₋₃. Direct C-implication in FC_1^* holds in the following cases (1) to (14), and only in these.

1 to 7 as in PC_1^* , see D_{26-1b}.

8 to 13 as in FC_1 , see D₂₈₋₂, but with 'Λ' instead of 'Λ' in (8) and (9).

14. $\{\mathfrak{S}_i(\mathfrak{A}_i)\}^* \xrightarrow{dC} \mathfrak{S}_i$.

Rule (14) refers to a transfinite conjunctive. Therefore, a rule of this kind can be established without the use of junctives by reference to a transfinite sentential class. In this way a transfinite rule corresponding to (14) was first proposed by Tarski (1927) and Hilbert (1931); see references in [Syntax] § 48, and further Tarski, *Journ. Symb. Log.*, vol. 4, 1939, p. 105. I have used a corresponding rule for language I ([Syntax] § 14, rule DC₂) and made more extensive use of transfinite rules also for variables of higher levels (rules of consequence for language II, [Syntax] § 34a-d, f); see also Rosser, *Journ. Symb. Log.*, vol. 2, 1937, p. 129.

The following syntactical theorems (T₁, 2, 3) will be used later for showing that, in every [L]-true interpretation for FC_1^* , the operators have an [L]-normal interpretation (T₃₁₋₁ and 2).

T₃₀₋₁.

a. If \mathfrak{S}_i (or \mathfrak{R}_i) $\xrightarrow{C} \mathfrak{S}_j$ in FC_1 , then \mathfrak{S}_i (or \mathfrak{R}_i^* , respectively) $\xrightarrow{C} \mathfrak{S}_j$ in FC_1^* . (From T_{23-41a}.)

b. If \mathcal{S}_i is C-true in FC_1 , it is C-true in FC_1^* . (From T23-41b.)

+T30-2. For any \mathcal{S}_i and i_p , with $\mathcal{R}_i = \{\mathcal{S}_i(i_p)\}$, the following holds in FC_1^* :

a. $(i_p)(\mathcal{S}_i) \xrightarrow{c} \mathcal{R}_i^*$.

b. $\mathcal{R}_i^* \xrightarrow{c} (i_p)(\mathcal{S}_i)$.

c. $(i_p)(\mathcal{S}_i)$ and \mathcal{R}_i^* are C-equivalent.

Proof. a. $(i_p)(\mathcal{S}_i)$ C-implies every element of \mathcal{R}_i (T28-4a, T1a), and therefore \mathcal{R}_i^* (T23-23). — b. $\mathcal{R}_i^* \xrightarrow{dc} \mathcal{S}_i$ (D3(14)). Therefore, $\mathcal{R}_i^* \xrightarrow{c} (i_p)(\mathcal{S}_i)$ (T28-4b, T1a, T23-3, T23-4). — c. From (a), (b).

+T30-3. Let i_p be the only free variable in \mathcal{S}_i , and let \mathcal{R}_i be $\{\mathcal{S}_i(i_p)\}$. Then the following holds in FC_1^* :

a. $\mathcal{R}_i^* \xrightarrow{c} (\exists i_p)(\mathcal{S}_i)$.

b. $(\exists i_p)(\mathcal{S}_i) \xrightarrow{c} \mathcal{R}_i^*$.

c. $(\exists i_p)(\mathcal{S}_i)$ and \mathcal{R}_i^* are C-equivalent.

Proof. a. Let \mathcal{M}_k be any class of junctives such that $\mathcal{R}_i^* \in \mathcal{M}_k$ and that the conditions (b), (c), and (d) in D23-4 are fulfilled. We have to show that $(\exists i_p)(\mathcal{S}_i) \in \mathcal{M}_k$. Since $\mathcal{R}_i^* \in \mathcal{M}_k$, at least one element of $\mathcal{R}_i \in \mathcal{M}_k$ (d); thus there is an i_m such that $\mathcal{S}_i(i_m) \in \mathcal{M}_k$. $\mathcal{S}_i(i_m) \xrightarrow{c} (\exists i_p)(\mathcal{S}_i)$ in FC_1 (T28-4c). Hence, $(\exists i_p)(\mathcal{S}_i) \in \mathcal{M}_k$ (T23-4od). — b. Let \mathcal{M}_k be any class of junctives such that $(\exists i_p)(\mathcal{S}_i) \in \mathcal{M}_k$ and that the conditions (b), (c), and (d) in D23-4 are fulfilled. We have to show that $\mathcal{R}_i^* \in \mathcal{M}_k$. Let \mathcal{S}_m be any closed sentence, and \mathcal{S}_n be $\text{disc}(\mathcal{S}_m, \text{neg}_C(\mathcal{S}_m))$. Then \mathcal{S}_n is C-true in PC_1 (T5-1a), hence in FC_1 (D28-2), hence $\mathcal{S}_n \in \mathcal{M}_k$ (T23-4of). $\mathcal{S}_n \xrightarrow{dc} \{\mathcal{S}_m, \text{neg}_C(\mathcal{S}_m)\}^v$ (D3(6)); hence this disjunctive belongs to \mathcal{M}_k (condition (b) for \mathcal{M}_k). Therefore, for any closed \mathcal{S}_m , either \mathcal{S}_m or $\text{neg}_C(\mathcal{S}_m) \in \mathcal{M}_k$ (condition (d)). Now we shall show that at least one sentence of \mathcal{R}_i belongs to \mathcal{M}_k . For the purpose of an indirect proof, let us suppose that no sentence of \mathcal{R}_i belonged to \mathcal{M}_k . Then, according to the result just found, for every sentence \mathcal{S}_l in \mathcal{R}_i , $\text{neg}_C(\mathcal{S}_l)$ would belong to \mathcal{M}_k , since \mathcal{S}_l is closed. Let \mathcal{R}_l be the class of these negations_C of the sentences of \mathcal{R}_i . Then \mathcal{R}_l^* would belong to \mathcal{M}_k (condition (c)). Let \mathcal{S}_j be $(i_p)(\text{neg}_C(\mathcal{S}_i))$. Then $\mathcal{R}_l^* \xrightarrow{dc} \mathcal{S}_j$ (D3(14)); hence \mathcal{S}_j would belong to \mathcal{M}_k (condition

(b)). On the other hand, $(\exists i_j)(\mathfrak{S}_i) \xrightarrow{c} \text{neg}_c(\mathfrak{S}_i)$ in FC_1 (T28-4d). Therefore, $\text{neg}_c(\mathfrak{S}_i) \in \mathcal{M}_k$ (T23-4od). Hence, $\{\mathfrak{S}_i, \text{neg}_c(\mathfrak{S}_i)\}^*$ would belong to \mathcal{M}_k (condition (c)). Every sentence is a C-implicate of $\{\mathfrak{S}_i, \text{neg}_c(\mathfrak{S}_i)\}$ in PC_1 (T5-21) and hence in FC_1 and, hence, would belong to \mathcal{M}_k (T23-4od) in contradiction to our supposition that no sentence of \mathfrak{R}_i belongs to \mathcal{M}_k . Therefore this supposition is false; at least one sentence of $\mathfrak{R}_i \in \mathcal{M}_k$. Hence $\mathfrak{R}_i \in \mathcal{M}_k$ (condition (d)). — c. From (a), (b).

T3b is especially noteworthy: an existential sentence C-implies the disjunctive of the instances of its operand. Thus we find a transfinite disjunctive as a C-implicate in FC_1^* , although the two disjunctive rules in PC_1^* refer only to finite disjunctives (with two and no elements respectively; see D26-1, rules 6 and 7) and no new disjunctive rule is added in FC_1^* (D3). This is brought about by the particular form of the definition of C-implication for junctives (D23-4).

Instead of the transfinite conjunctive rule for the universal operator in FC_1^* (D3, rule 14), we could use the following transfinite disjunctive rule for the existential operator.

Rule 14'. $(\exists i_k)(\mathfrak{S}_i) \xrightarrow{ac} \{\mathfrak{S}_i^{(i_k)}\}$ where i_k is the only free variable in \mathfrak{S}_i . (As to the reason for the restricting condition, see, below, remark on D31-2.)

Rule (14') leads to the same results as rule (14) (that is to say, FC_1^* and the calculus containing (14') instead of (14) are coincident calculi [I] D31-9).

§ 31. FC_1^* is a Full Formalization of Functional Logic

In any $[L-]$ true interpretation for FC_1^* , the universal operator has an $[L-]$ normal interpretation (T_1); that is to say, a universal sentence is $[L-]$ equivalent to the conjunctive of the instances of its operand (D_1). Likewise, in any $[L-]$ true interpretation for FC_1^* , the existential operator has an $[L-]$ normal interpretation (T_2); that is to say, an existential sentence is $[L-]$ equivalent to the disjunctive of the instances of its operand (D_2). Hence, FC_1^* is a full formalization of functional logic.

As we have said earlier (§ 28), we presuppose a system of functional logic of such a kind that every individual in it is directly designated. Therefore, a universal sentence $(i_k)(\mathfrak{S}_i)$ is true if and only if every instance of \mathfrak{S}_i is true. Hence, if we use junctives, the universal sentence is true if and only if the conjunctive of the instances of \mathfrak{S}_i is true; both are L -equivalent to one another. Analogously, the existential sentence $(\exists i_k)(\mathfrak{S}_i)$ (if it is closed) is true if and only if at least one instance of \mathfrak{S}_i is true; it is therefore L -equivalent to the disjunctive of the instances. On the basis of these considerations, we can define the concepts of normal interpretations of the operators (D_{1a} , D_{2a}) with respect to FC_1^* . These definitions are simpler than the former ones with respect to FC_1 (D_{28-6} and 7). It is easy to see (with the help of D_{21-3} and 4) that the new concepts are in accordance with the previous ones. Further, the concepts of L -normal interpretations are here easily definable (D_{1b} , D_{2b}).

+**D31-1a** [**b**]. The **universal operator** in a calculus K (containing a form of FC or FC^* with junctives) has an **$[L-]$ normal interpretation** in $S =_{Df} S$ is an $[L-]$ true interpretation for K , and for every \mathfrak{S}_i and i_k in K such that i_k is the only free variable in \mathfrak{S}_i , $(i_k)(\mathfrak{S}_i)$ is $[L-]$ equivalent to $\{\mathfrak{S}_i(i_k)\}^*$ in S .

+**D31-2a** [**b**]. The **existential operator** in a calculus K

(containing a form of FC or FC^{*} with junctives) has an [L-]normal interpretation in $S =_{D1} S$ is an [L-]true interpretation for K , and for every \mathfrak{S}_i and i_k in K such that i_k is the only free variable in \mathfrak{S}_i , $(\exists i_k)(\mathfrak{S}_i)$ is [L-]equivalent to $\{\mathfrak{S}_i(i_k)\}^*$ in S .

The following counter-example shows that the condition that i_k is the only free variable in \mathfrak{S}_i is essential for D2. In functional logic, $'(\exists x)R(x,y)'$ is not L-equivalent to the disjunctive of the instances $'R(a,y)'$, $'R(b,y)'$, etc. The existential sentence is L-equivalent to $'(y)(\exists x)R(x,y)'$, while the instances are L-equivalent to $'(y)R(a,y)'$, $'(y)R(b,y)'$, etc., respectively, and hence their disjunctive is L-equivalent to $'(\exists x)(y)R(x,y)'$. This sentence is stronger than $'(y)(\exists x)R(x,y)'$.

The same condition in D1 is not essential (the proof for T1 makes no use of it) but has been added merely for the sake of analogy.

+T31-1a [b]. If S is an [L-]true interpretation for FC₁^{*}, then the universal operator in FC₁^{*} has an [L-]normal interpretation in S .

Proof for a [b]. Let S be an [L-]true interpretation for FC₁^{*}. Then, for any \mathfrak{S}_i and i_k in FC₁^{*}, $(i_k)(\mathfrak{S}_i)$ and $\{\mathfrak{S}_i(i_k)\}^*$ are C-equivalent in FC₁^{*} (T30-2c) and hence [L-]equivalent in S (T23-18 [19c]). Thus the universal operator has an [L-]normal interpretation in S (D1).

It is easy to see that a transfinite rule is necessary in order to assure the [L-]normal interpretation of the operators in a calculus K containing FC₁. $(i_k)(\mathfrak{S}_i)$ and $\{\mathfrak{S}_i(i_k)\}^*$ must be C-equivalent in K . The rules of FC₁ suffice to make every instance of \mathfrak{S}_i , and hence also their conjunctive, a C-implicate of $(i_k)(\mathfrak{S}_i)$. The problem is how to make the universal sentence a C-implicate of the conjunctive of instances. The universal sentence is not an L-implicate of any proper sub-class of the conjunctive, since from the fact that some individuals have a certain property we cannot infer that all have it; still less is it an L-implicate of any finite sub-class. Here, therefore, a transfinite rule is necessary which makes

use of the whole transfinite class of instances, as rule (14) in FC_1^* (D30-3) does and the alternative rule (14') mentioned above.

+**T31-2a [b]**. If S is an $[L]$ -true interpretation for FC_1^* , then the existential operator in FC_1^* has an $[L]$ -normal interpretation in S .

Proof for a [b]. Let S be an $[L]$ -true interpretation for FC_1^* . For any \mathfrak{S}_i and i_k in FC_1^* such that i_k is the only free variable in \mathfrak{S}_i , $(\exists i_k)(\mathfrak{S}_i)$ and $\{\mathfrak{S}_i(i_k)\}^\vee$ are C-equivalent in FC_1^* (T30-3c) and hence $[L]$ -equivalent in S (T23-18 [19c]). Thus the existential operator has an $[L]$ -normal interpretation in S (D2).

T1 and 2 show that FC_1^* is a *full formalization of functional logic*.

An existential sentence can be transformed in FC_1 into a C-equivalent sentence with a universal operator and two signs of negation_C (T28-4d, e). Therefore, if the universal operator and the sign of negation_C have an $[L]$ -normal interpretation, then the same holds for the existential operator. On the other hand, a universal sentence can be transformed into a C-equivalent sentence with an existential operator and two signs of negation_C (T28-4f, g). Therefore, if the existential operator and the sign of negation_C have an $[L]$ -normal interpretation, then the same holds for the universal operator. Thus we have seen that in FC_1^* (D30-3), where neg_C has always an $[L]$ -normal interpretation because of the subcalculus PC_1^* (T27-1), the rule (14) for the universal operator suffices to assure the $[L]$ -normal interpretation not only for this operator (T1) but also for the existential operator (T2). Likewise, the rule (14') for the existential operator (see § 30 at the end), taken instead of (14), would suffice to assure the $[L]$ -normal interpretation for both operators.

§ 32. Involution

An alternative to the use of junctives is outlined. It consists in the introduction of the concept of involution (D_1) and the corresponding L- and C-concepts (D_2 and 6). A calculus FC_1^{**} is given in the form of a definition for 'direct C-involution' (D_{12}). This calculus corresponds to FC_1^* ; however, it refers not to junctives but only to sentences and sentential classes. FC_1^{**} is, like FC_1^* , a full formalization for functional logic.

An alternative to the use of junctives will briefly be explained here, a semantical and syntactical terminology which allows the formulation of the same things we have formulated above in terms of junctives.

We have previously introduced junctives in syntax in connection with the concept of C-implication (§ 23). We have seen that the reference to a conjunctive \mathfrak{R}_i is essential only when it occurs as a C-implicans, while its occurrence as a C-implicate can always be avoided by a reference to the sentences of \mathfrak{R}_i . On the other hand, the reference to a disjunctive \mathfrak{R}_i^v is essential only when it occurs as a C-implicate. This suggests the introduction of a term, say 'C-involution', for the special case of the relation of C-implication between a conjunctive and a disjunctive. Therefore, we shall introduce 'involution' (D_1) in such a way that ' \mathfrak{R}_i involves \mathfrak{R}_j ' means the same as previously ' \mathfrak{R}_i implies \mathfrak{R}_j '; the terms 'L-involves', 'F-involves', and 'C-involves' will be used in an analogous way. However, 'involves' will not be defined in terms of junctives. We shall use it in a metalanguage which does not refer to junctives but only to sentences and neutral sentential classes. These classes are neutral in the sense that they are construed neither conjunctively nor disjunctively. Therefore, the concept of truth is not applied to sentential classes but only to sentences. This concept is taken here as

basic (in D_1); the other radical concepts may be defined as previously ([I] § 9) but with respect to sentences only.

D32-1. \mathfrak{R}_i involves \mathfrak{R}_j (\mathfrak{R}_j is an involute of \mathfrak{R}_i ; $\mathfrak{R}_i \rightarrow \mathfrak{R}_j$) (in S) = D_1 at least one sentence of \mathfrak{R}_i is not true or at least one sentence of \mathfrak{R}_j is true.

We define this and the following concepts with respect to sentential classes only. We make the general convention that the application of one of these concepts to a sentence \mathfrak{S}_i is an abbreviation for its application to $\{\mathfrak{S}_i\}$.

T32-1. $\mathfrak{S}_i \rightarrow \mathfrak{S}_j$ if and only if $\mathfrak{S}_i \rightarrow \mathfrak{S}_j$.

Proof. $\mathfrak{S}_i \rightarrow \mathfrak{S}_j$ if and only if $\{\mathfrak{S}_i\} \rightarrow \{\mathfrak{S}_j\}$ (convention), hence if and only if \mathfrak{S}_i is false or \mathfrak{S}_j is true (D_1), hence if and only if $\mathfrak{S}_i \rightarrow \mathfrak{S}_j$, ([I] D9-3).

On the basis of D_1 and T_1 , 'involution' can now be applied to \mathfrak{T} , i.e. to members which are either sentences or (neutral) sentential classes.

The concept of L-involution could be introduced either by a reformulation of the postulates for L-concepts (§ 22, [I] § 14) or on the basis of the concept of the L-range of a sentence ($Lr\mathfrak{S}_i$, §§ 11 and 22, [I] § 20). We shall indicate here the second way. L-implication corresponds to inclusion of L-ranges (D_{11-7}); hence $\mathfrak{R}_i^* \xrightarrow{L} \mathfrak{R}_j^*$ if and only if the product of the L-ranges of the sentences of \mathfrak{R}_i is contained in the sum of the L-ranges of the sentences of \mathfrak{R}_j (D_{22-1} and 2). This leads to D_2 .

D32-2. \mathfrak{R}_i L-involves \mathfrak{R}_j (\mathfrak{R}_j is an L-involute of \mathfrak{R}_i ; $\mathfrak{R}_i \xrightarrow{L} \mathfrak{R}_j$) (in S) = D_1 the product of the L-ranges of the sentences of \mathfrak{R}_i is contained in the sum of the L-ranges of the sentences of \mathfrak{R}_j .

In the syntax of junctives, the rules of deduction of a calculus K are formulated as a definition of 'direct C-implication in K ' (§ 23). On the basis of this concept, C-implica-

tion (D23-4) and the other C-concepts are defined in such a way that they fulfill the requirement of adequacy, i.e. that they hold in all those cases, and only those, in which the corresponding radical concepts hold in every true interpretation for K . An analogous procedure can be applied for the introduction of 'C-involution'. Here, the rules of deduction define 'direct C-involution' (' \dashv_{dc} '). We have to begin with a definition of 'true interpretation', analogous to D23-1.

D32-5a [b]. S is an $[L]$ -true interpretation for $K =_{Df}$ S is an interpretation for K ($[I]$ D33-1), and for every \mathfrak{X}_i and \mathfrak{X}_j , if $\mathfrak{X}_i \dashv_{dc} \mathfrak{X}_j$ in K , $\mathfrak{X}_i \dashv_{[L]} \mathfrak{X}_j$ in S .

The definition of 'C-involution' is analogous to D23-4 but simpler.

D32-6. \mathfrak{R}_i C-involves \mathfrak{R}_j (\mathfrak{R}_j is a C-involute of \mathfrak{R}_i ; $\mathfrak{R}_i \dashv_c \mathfrak{R}_j$) (in K) = Df every class \mathfrak{R}_k which fulfills the following conditions, (a) and (b), contains at least one sentence of \mathfrak{R}_j .

a. $\mathfrak{R}_i \subset \mathfrak{R}_k$.

b. For every \mathfrak{R}_m and \mathfrak{R}_n , if $\mathfrak{R}_m \subset \mathfrak{R}_k$ and $\mathfrak{R}_m \dashv_{dc} \mathfrak{R}_n$, then at least one sentence of $\mathfrak{R}_n \in \mathfrak{R}_k$.

\mathfrak{R}_k in D6 corresponds to \mathfrak{M}_k in D23-4. In analogy to T23-1, it can here easily be seen that, if \mathfrak{R}_k is the class of the sentences in K which are true in a true interpretation S for K , then \mathfrak{R}_k fulfills the condition (b) in D6. Further, in analogy to T23-11: If S is an $[L]$ -true interpretation for K and $\mathfrak{X}_i \dashv_c \mathfrak{X}_j$ in K , then $\mathfrak{X}_i \dashv_{[L]} \mathfrak{X}_j$ in S . Hence D6 fulfills the requirement of adequacy. The same holds for D7 and 8.

D32-7. \mathfrak{S}_i is C-true (in K) = Df $\Lambda \dashv_c \mathfrak{S}_i$.

D32-8. \mathfrak{S}_i is C-false (in K) = Df $\mathfrak{S}_i \dashv_c \Lambda$.

Any sentence or rule in the metalanguage (semantics or syntax) formulated in terms of junctives can easily be translated into a sentence or rule formulated in terms of in-

volution. For instance, a sentence stating the relation of implication (or L-implication, or C-implication, respectively) between two junctives is translated into a sentence stating the relation of involution (or L-involution, or C-involution, respectively) in the following way. ' \mathfrak{S}_i ' remains unchanged; ' \mathfrak{R}_i^* ' as (L-, C-) implicans and ' \mathfrak{R}_i^v ' as (L-, C-) implicate are replaced by ' \mathfrak{R}_i '; ' \mathfrak{R}_i^* ' as (L-, C-) implicate and ' \mathfrak{R}_i^v ' as (L-, C-) implicans are replaced by 'every sentence of \mathfrak{R}_i '.

As an example of the formulation of the rules of deduction of a calculus K as a definition for 'direct C-involution in K ', we shall state the rules for the calculus FC_1^{**} . This calculus corresponds to FC_1^* (D30-3) in the sense that its rules result if we translate the rules of FC_1^* from the syntax language of junctives into the syntax language of involution in the way just indicated. Therefore the calculus FC_1^{**} is likewise a *full formalization of functional logic*.

D32-12. Direct C-involution in FC_1^{}** holds in the following cases, (1) to (14), and only in these.

- 1 to 5 as in D2-2b but with ' \vdash_{dC} ' instead of ' \overrightarrow{dC} '.
6. $\text{dis}_C(\mathfrak{S}_i, \mathfrak{S}_j) \vdash_{dC} \{\mathfrak{S}_i, \mathfrak{S}_j\}$, where \mathfrak{S}_i and \mathfrak{S}_j are closed.
7. $\vee \vdash_{dC} \wedge$.
- 8 to 13, as in D28-2, but with ' \vdash_{dC} ' instead of ' \overrightarrow{dC} '.
14. $\{\mathfrak{S}_i(i_k)\} \vdash_{dC} \mathfrak{S}_i$.

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